THEORETICAL INVESTIGATIONS OF
BOUNDARY LAYER STABILITY

GIBBS S. RAETZ and W. BYRON BROWN
NORTHROP CORPORATION

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FOREWORD

This report was prepared by Gibbs S. Raetz and W. Byron Brown of the Boundary Layer Research Section under the direction of Dr. Werner Penninger, Northrop Norair, a Division of Northrop Corporation, Hawthorne, California, and covers research investigations performed from July 1963 to August 1964. This work was performed under Air Force Contract AF33(657)-11618, Project Number 1366, Task Number 136612, "Application of Laminar Flow Control Technology to Optimum Supersonic Cruise."

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ABSTRACT

The mathematical analysis underlying a Fortran program for calculating the proper solutions of the Orr-Sommerfeld system with sufficient accuracy and economy for applying the resonance theory of transition is described. This program covers spacewise growths, rather than timewise growths as in previous computations, of mainly two-dimensional Fourier components of the motion. It employs various innovations providing as much accuracy from efficient single-precision arithmetic as would be obtained from awkward multiple-precision arithmetic in previous calculation schemes. The source programs and some sample calculations, for the principal mode of oscillation of the Blasius boundary layer, are included.

The Lee-Lin stability equations for compressible flow have been extended to include the terms involving the component of the mean boundary layer flow perpendicular to the flat plate. At Mach 5 this more than doubled the critical Reynolds number. Allowance was then made for the three-dimensional aspect of the disturbance velocity. The final result was to give good agreement with observed data in the lower branch of the neutral stability curve at Mach 2.2 and Mach 5, fair agreement with the upper branch at Mach 2.2 and large discrepancies with the data in the upper branch at Mach 5.

Comparison of experimentally determined neutral stability curves with those computed by simplified approximations have disagreed considerably at high Mach numbers on the upper branch, even when agreement was fairly good on the lower branch. To improve the calculations, the complete set of three-dimensional stability equations, including all three momentum equations and also the component of the mean flow in the boundary layer normal to the surface, are solved numerically. This set of equations can be reduced to a set of eight linear equations with complex coefficients. The theoretical solutions for Mach 2.2 and Mach 3 are compared with experimental data and show good agreement in both upper and lower branches.
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PART I

CALCULATION OF PRECISE PROPX SOLUTIONS
FOR THE RESONANCE THEORY OF TRANSITION

Gibbs S. Eastz
BASIC NOTATION

a frequency ratio (streamwise/principal)
c timewise frequency
f_m amplitude coefficient (canonical)**
E_m amplitude coefficient ( logarithmic)**
b_m amplitude coefficient (global)**
i unit imaginary number
p amplitude coefficient (pressure)
s metrical coefficient denominator (local)
e_n expansion coefficient of metrical coefficient denominator (local)***
t metrical coefficient numerator (local)
e_n expansion coefficient of metrical coefficient numerator (local)***
u amplitude coefficient (principal velocity component)
u_{i,j} amplitude coefficient (original velocity component)*
w amplitude coefficient (normal velocity component)
x coordinate (local)
x_{i,j} coordinate (Cartesian)
y coordinate (global)
y_{o} origin (local)
z coordinate (normal)

* l = 1, 2, 3 (terms summed over this range where same index appears twice)
** n = 1, 2, 3, 4
*** n = 0, 1, ..., 3M (M given)
\( k_1 \) spacewise frequency

\( F_m \) amplitude coefficient (asymptotic)**

\( l \) formal series degree

\( R \) Reynolds number (local)

\( U \) basic flow velocity component (principal)

\( U_d \) basic flow velocity component (original)**

\( \omega \) principal frequency

\( \beta \) frictional frequency

\( \gamma \) phase velocity parameter (local)

\( \delta \) expansion radius (local)

\( \Delta_m \) secular determinant (local)**

\( k \) metrical coefficient (global)

\( \lambda \) metrical coefficient (local)

\( \mu \) viscosity (shearing)

\( f_m \) fundamental solution (local)**

\( p \) density

\( \delta \) Reynolds number parameter (local)

\( \tau \) phase velocity

\( \eta_m \) amplitude coefficient (local)**

\( \Gamma \) frictional parameter

\( \Theta \) secular determinant (global)

\( A \) reference length

\( T \) reference velocity

---

* \( d = 1,2,3 \) (terms summed over this range where same index appears twice)

** \( n = 1,2,3,4 \)
\( \wedge \) dimensionless value
\( \sim \) complex conjugate value
\( (\cdot)\) total derivative
\( (\cdot)^m \) fundamental solution**

** m = 1, 2, 3, 4

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I. INTRODUCTION

The resonance theory of transition is a non-linear theory of boundary layer oscillations intended to explain the principal motions during transition and to provide a practical technique for calculating such motions. Its present form is described in detail in Reference 1.

This theory was deduced from the continuity and Navier-Stokes equations using a decomposition of the whole motion into a perturbation series and then a decomposition of each perturbation into a Fourier spectrum with aperiodically varying coefficients. The first perturbation is produced by boundary irregularities such as wall waviness or suction uneveness and external turbulence or sound. The higher perturbations are generated by a coupling with lower perturbations through the non-linear terms in the Navier-Stokes equations. Each Fourier component of the higher perturbations is driven by a pair of Fourier components from lower perturbations in such a way that a partial resonance can occur in certain situations. During the stronger partial resonances, the driven Fourier component undergoes a rapid and large growth in amplitude and a gradual shift in phase. Otherwise, it somewhat follows a trend indicated by linear stability theory. Altogether, the re-composed motion is distinctly non-linear and, just as has been observed, should explain various transition phenomena beyond the scope of previous concepts.

In calculating the motion, the first perturbation spectrum is to be ascertained directly from the boundary irregularities, and the higher perturbation spectra are to be estimated recursively from lower perturbation spectra. For such calculations, precise proper solutions of the actual and adjoint Orr-Sommerfeld systems or their generalizations are essential. The actual and adjoint proper functions are required to evaluate the coupling between the driver and driving Fourier components, and the proper values are needed to determine the amplitude and phase modulations of the driven Fourier component. Once such data are readily obtainable, the estimation and analysis of various important phenomena should become relatively easy.

Thus, as an initial step in the implementation of the theory, a Fortran program for calculating the proper solutions of the actual Orr-Sommerfeld system with sufficient accuracy and efficiency was developed. The underlying mathematical analysis together with the program and some sample calculations are presented here.

In this particular program, to minimize distractions by secondary details, just a flat wall, a two-dimensional basic flow, and an incompressible fluid are considered. Also, only two-dimensional perturbations with waves traveling in the same direction as the basic flow are covered. However, the proper values for three-dimensional perturbations with waves traveling in all directions can be ascertained and, after minor extensions, the corresponding proper functions could be obtained as well. Contrary to custom, spacewise rather than time-wise modulations of the Fourier components are allowed, since only the spacewise variations are important in most actual problems. Besides the principal mode of oscillation usually considered in stability theory, some higher modes also can be investigated.
In the sample calculations, a set of proper values which also could be used in linear stability theory and some typical proper functions are obtained. The Blasius basic flow, two-dimensional perturbations, and the principal modes are considered.

Previous methods of solving the Orr-Sommerfeld system such as those described in References 2 thru 4, which were developed for ordinary linear stability theory, were not entirely appropriate for the present application. As one example, the method of asymptotic expansions (References 2, 3, and 4) neglects all but the first term of an expansion which probably diverges, and it allows some excessive errors in that term. As another example, the method of numerical integration (Reference 5) can entail a large accumulative error, due to a spurious solution which enters the numerical solution through truncation errors and then tends to grow excessively. As a further example, the previous methods generally pertain to time-wise rather than space-wise growths of Fourier components.

The present method avoids such inadequacies and also utilizes some innovations which further improve the precision. For example, instead of finding the secular determinant from extremely slight differences between fundamental solutions as in most previous methods, it employs a supplementary differential system for the secular determinant. As a result, it needs just single-precision arithmetic for calculations which in other schemes would require awkward multiple-precision arithmetic. Analogous innovations should be especially helpful in similar calculations for supersonic and hypersonic boundary layers, where the precision is even more critical. They should reduce the computing cost substantially in some cases and enable otherwise impractical or impossible analyses in other cases.

In developing and applying the program, the author was aided by Mr. Lester Pickett and Mrs. Dorothy McHugh, whose assistance is acknowledged with gratitude.
II. DIFFERENTIAL SYSTEMS

A. OHR-SOMMERFELD

The basic objective is to solve the approximate differential system for resonance amplitude coefficients deduced in Reference 1. Starting from Equations (114) and (115) of that reference and omitting superscript indices, this system may be expressed as

\[ iu_j A_j + u'_j = 0 \]
\[ ip_j (A_k C_k + c) + p u'_j u_k + i p k A_j = \mu (u_j C_k - u_k A_j) \]  
\[ (p u_j A_k C_k + c) + p' = \mu (u_j C_k - u_k A_j) \]  

and

\[ u_j (0) = u_j (0) = 0 \]
\[ u_j (\omega) = u_j (\omega) = p (\omega) = 0 \]

where \( j, k = 1, 2 \) and the primes denote ordinary derivatives with respect to \( x_3 \). Here, Cartesian coordinates \( x_j (j = 1, 2, 3) \) have been used, with the wall surface at \( x_3 = 0 \) and the adjoining flow at \( x_3 = \omega \). The basic flow velocity components \( U_j \) are to be given, while the resonance amplitude coefficients \( u_j \) and \( p_j \) of the perturbation velocity components and pressure, respectively, are to be sought. The density \( \rho \) and viscosity \( \mu \) are known constants, whereas the spacewise frequencies \( A_j \) and timewise frequency \( c \) are partly unknown parameters. Also, \( A_j \) has been eliminated from the system by a transformation described in Part G-3 of Reference 1, leaving just the value \( A_j = 0 \) to be considered here. Consequently, if \( A_j \) really is non-zero and the actual coefficients are needed, but not otherwise, the inverse of that transformation must be applied to the solution. Dimensionless quantities are chosen as

\[ \hat{\rho} = \rho / \rho = 1 \]
\[ \hat{\mu} = \mu / \rho T = 1 / R \]
\[ \hat{A}_j = A_j R \]
\[ \hat{c} = c / T \]
\[ \hat{x}_j = x_j / R \]
\[ \hat{u}_j = u_j / R \]
\[ \hat{p}_j = p_j / R T \]

with the result that, disregarding the overscript, Equations (1) and (2) apply directly to dimensionless as well as dimensional quantities.
For definiteness, the basic flow is taken in the $x_2$-direction, so that $\hat{U}_2$ vanishes, and it necessarily is regarded as unseparated. The reference length $\Lambda$ may be any characteristic thickness of the boundary layer, but for the Blasius basic flow considered in the sample calculations it is chosen as $x_3/R$. The reference velocity $\tilde{U}$ is chosen as the adjoining flow velocity, so that $\tilde{U}_1$ varies from 0 to 1 monotonically as $x_3$ varies from 0 to $\infty$. The streamwise frequency $\hat{A}_1$ is to be complex, whereas the crosswise frequency $\hat{A}_2$ and the timewise frequency $\hat{D}$ are to be real. Otherwise, only dimensionless quantities with the overscrpt omitted are considered further.

For present purposes, an alternate form of the differential system is preferred, which involves the additional parameters

\[
\begin{align*}
\sigma &= (\Lambda_1 A_2)^{1/2} \\
a &= \Lambda_1 / \sigma \\
b &= A_2 / \sigma \\
\tau &= -c / \Lambda_1 
\end{align*}
\]

with $Re(\omega) > 0$ and the additional variables

\[
\begin{align*}
z &= x_3 \\
v &= \tilde{U}_1 \\
u &= A_2 \tilde{u}_1 / \sigma \\
w &= \tilde{u}_3 \\
\omega &= iA_2 \tilde{u}_1 - iA_2 \tilde{u}_2
\end{align*}
\]

Substituting these in Equations (1) and (2), the alternate system is obtained as

\[
\begin{align*}
1 & \sigma \omega u' = 0 \\
1 & \sigma \omega (U - \tau)u + \sigma \omega U'v + i\sigma \omega \mu(u'' - \sigma^2 u) \\
1 & \sigma \omega (U - \tau)w + \omega w' = \mu(w'' - \sigma^2 w) \\
1 & \sigma \omega (U - \tau)u + i\sigma \omega \mu w' = \mu(u'' - \sigma^2 u)
\end{align*}
\]

and

\[
\begin{align*}
u(0) &= \tilde{v}(0) = \tilde{u}(0) = 0 \\
\omega(\infty) &= \tilde{v}(\infty) = \tilde{u}(\infty) = \mu(\infty) = 0
\end{align*}
\]

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Clearly, the first three of Equations (6) may be solved for $u$, $w$, $\rho$ independently of the fourth. As a result, the first three suffice to determine the proper values and functions for two-dimensional perturbations (with $A_2 = 0$) and also the proper values for three-dimensional perturbations (with $A_2 \neq 0$), which are the quantities sought here. Hence, the fourth is not considered further.

To reduce the order of the remaining system, $u$ is eliminated, yielding

$$w'' - \alpha^2 w' = i\rho\Re(a(U - \tau)w' - aU'w - i\omega\rho)$$
$$p' = -i\rho\Re((U - \tau)w + \mu(iw'' - \alpha^2 w))$$

and

$$w(0) = w'(0) = 0$$
$$w(\omega) = p(\omega) = 0$$

Elimination of $p$ from this system yields the customary form of the Orr-Sommerfeld system

$$w'' - \alpha^2 w' + \alpha^4 w = i\rho\Re((U - \tau)(w'' - \alpha^2 w) - U'w)$$

and

$$w(0) = w'(0) = 0$$
$$w(\omega) = w'(\omega) = 0$$

(where $\rho = a = 1$) which however is not needed here.

To obtain a first-order set of equations, the new variables

$$\xi_1 = w$$
$$\xi_2 = w'$$
$$\xi_3 = w'' - \alpha^2 w$$
$$\xi_4 = \alpha^2 \rho p$$

are introduced, whereupon Equations (8) and (9) become

$$\xi_1' = \xi_2$$
$$\xi_2' = \xi_3 + \alpha^2 \xi_1$$
$$\xi_3' = \xi_4 + i\rho\Re[(U - \tau)\xi_2 - U'\xi_1]$$
$$\xi_4' = \alpha^2 \xi_3 - i\rho\Re(U - \tau)\xi_1$$

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and

\[ f_1(x) = f_2(x) = 0 \]
\[ f_3(x) = f_4(x) = 0 \]  (14)

which are the basis for the ensuing analysis.

Outside the basic flow, as \( z \) approaches \( \infty \) and \( U' \) becomes negligible, Equations (13) approach the asymptotic system

\[ f_1' = f_2 \]
\[ f_2' = f_3 + \alpha^2 f_1 \]
\[ f_3' = f_4 + i\alpha \omega (1 - \gamma)f_2 \]
\[ f_4' = \alpha^2 [f_3 - i\alpha \omega (1 - \gamma)f_1] \]  (15)

with constant coefficients. This has the four fundamental solutions

\[ f_1^{(1)} = \exp(-\beta z) \]
\[ f_1^{(2)} = \exp(-\alpha z) \]
\[ f_1^{(3)} = \exp(\alpha z) \]
\[ f_1^{(4)} = \exp(\beta z) \]  (16)

where

\[ \beta = [\alpha^2 - i\alpha \omega (1 - \gamma)]^{1/2} \]  (17)

with \( \text{Re}(\beta) > 0 \). The corresponding general solution is

\[ F_1 = \sum_{m=1}^{\infty} B_m p_m^{(m)} \]  (18)

where each \( B_m \) is an arbitrary constant. However, to conform with Equations (16), the restriction \( B_3 = B_4 = 0 \) is necessary, so that the corresponding complete solution is

\[ F_1 = B_1 f_1^{(1)} + B_2 f_1^{(2)} \]  (19)

Inside the basic flow, where \( U' \) is significant, Equations (13) have four fundamental solutions \( f_1^{(m)} \) which approach \( f_1^{(m)}(\alpha) \) as \( z \) approaches \( \infty \) (m = 1, 2, 3, 4). Thus, in that region, the general solution is

\[ f_1' = \sum_{m=1}^{4} B_m f_1^{(m)} \]  (20)
and the applicable complete solution is

\[ f_1 = B_1 f_1^{(1)} + B_2 f_1^{(2)} \]  \hspace{1cm} (21)

Across much of the basic flow, at the more important conditions, both \( f_1^{(1)} \) and \( f_1^{(2)} \) tend to vary rapidly and greatly. Meanwhile, toward the wall, at least in the Mauve basic flow, \( f_1 \) tends to become an extremely slight difference between components of those functions. These and other properties greatly hinder actual calculations unless, as done here, special procedures are employed.

At the wall, where \( z = 0 \), to comply with Equations (16), the two quantities

\[ f_1 = B_1 f_1^{(1)} + B_2 f_1^{(2)} \]
\[ f_2 = B_1 f_2^{(1)} + B_2 f_2^{(2)} \]  \hspace{1cm} (22)

must vanish simultaneously, where \( f_1^{(1)} \) and \( f_1^{(2)} \) are the first derivatives of \( f_1^{(1)} \) and \( f_1^{(2)} \), respectively, with respect to \( z \). But in a non-trivial solution this can happen only if the determinant of the system

\[ 0 = f_1^{(1)} f_2^{(2)} - f_1^{(2)} f_2^{(1)} \]  \hspace{1cm} (23)

called the secular determinant vanishes there. Quite obviously, the secular determinant is a function of the parameters of Equations (13). Thus, at \( z = 0 \), the secular equation

\[ 6(A_1, A_2, c, R) = 0 \]  \hspace{1cm} (24)

must be satisfied, which determines the proper value of \( A_1 \) as an implicit function of \( A_2, c, R \). This value is not unique but instead ranges over a sequence of discrete values called the proper value spectrum. The corresponding complete solutions are called the proper function spectrum, and both spectra together are called the proper solution spectrum. Usually only the proper value with the algebraically smallest imaginary part and the associated proper function, called the principal proper solution, are needed. In linear stability theory, this solution would represent the most unstable and therefore the principal mode of oscillation, and the corresponding variation of \( \text{Re}(A_1) \) with \( R \) at \( \text{Im}(A_1) = A_2 = 0 \) would be the customary neutral curve.

For the sample calculations, wherein \( A_2 = 0 \), the ranges of \( c \) and \( R \) are chosen as

\[
\begin{align*}
-0.005 & \leq c \leq -0.100 \\
125 & \leq R \leq 2500.
\end{align*}
\]  \hspace{1cm} (25)

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so as to cover the main regions of most actual transitions. Consequently, the most difficult condition concerned here is at $c = -100$ and $\lambda = 2900$, where a proper value like $A_1 = 0.75 + 10.03$ and therefore values like

$$
\sigma = 0.25 + 10.03 \\
\beta = 19 + 12
$$

are encountered.

At this condition, to illustrate an obstacle which often has been overlooked, suppose that the method of numerical integration is tried. For simplicity, assume that $A_1$ and $f_{\infty}^{(0)}$ in $\pi = 1, 2, 3, 4$ have been ascertained in some way and that an integration for $f_1$ across the basic flow, from $z = 0$ to approximately $z = 5$, is sought. According to Equation (21), $f_1$ correctly contains components of only $f_{\infty}^{(1)}$ and $f_{\infty}^{(2)}$, which vanish as $z$ becomes infinite.

But as the integration proceeds, truncation errors unavoidably will occur, introducing spurious components of $f_{\infty}^{(3)}$ and $f_{\infty}^{(4)}$ into the numerical solution. Initially these unwanted components may be very small, but eventually they can become excessively large, since both $f_{\infty}^{(3)}$ and $f_{\infty}^{(4)}$ grow unboundedly as $z$ becomes infinite. In particular, the components of $f_{\infty}^{(4)}$ finally will grow like $f_{\infty}^{(2)}$ does, so that the ratios of the final to the initial error magnitudes for $0 \leq z \leq 5$ will tend to resemble $|f_{\infty}^{(4)}(5)| = (10)^{6/2}$ in magnitude.

Consequently, unless extraordinarily precise arithmetic is used, the numerical solution will become meaningless before the integration is completed, although it deceptively may remain smooth enough to appear accurate. Furthermore, even if the necessary precision were provided, the integration then would be too cumbersome and costly to be really practical. For such reasons, a more sophisticated form of the differential equations, allowing a more dependable and efficient method of solution, is deduced and applied here.

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3. FUNDAMENTAL SOLUTION

Thus, to overcome various obstacles, \( f_m \) \((m = 1, 2, 3, 4)\) are replaced by the new variables:

\[
\begin{align*}
g_1 & = \log(f_1) \\
g_2 & = f_2/f_1 \\
g_3 & = f_3/f_1 \\
g_4 & = f_4/f_1 \\
\end{align*}
\]

whereupon Equations (13) transform to

\[
\begin{align*}
g_1' & = g_2 \\
g_2' & = -g_2g_2 + g_3 + c^2 \\
g_3' & = -g_2g_3 + g_4 + i\omega_0 \mathcal{R}(U - 1)g_2 - U' \\
g_4' & = -g_2g_4 + \alpha [g_3 - i\omega_0 \mathcal{R}(U - 1)] \\
\end{align*}
\]

Here the last three equations can be solved for \( g_2, g_3, g_4 \) independently of the first equation, which for simplicity is not considered further.

The remaining equations are to be integrated from \( z = 0 \) to \( z = z_0 \), for two fundamental solutions \( g_{2(1)} \) and \( g_{2(2)} \), corresponding to \( f_{(1)} \) and \( f_{(2)} \), respectively, which approach \( F_{(1)} \) and \( F_{(1)} \) as \( z \) approaches \( \infty \). Hence, complying with Equations (15) and (15), the initial values at \( z = 0 \) for the two solutions are taken as

\[
\begin{align*}
g_{2(1)}(0) & = \beta \\
g_{2(2)}(0) & = -\alpha \\
g_{3(1)}(0) & = 1 \\
g_{3(2)}(0) & = 0 \\
g_{4(1)}(0) & = 0 \\
g_{4(2)}(0) & = \sigma \tau \\
\end{align*}
\]

where

\[
\tau = i\omega_0 \mathcal{R}(1 - r)
\]

Also, \( \beta \) is replaced by the re-normalized secular determinant

\[
\Theta = \delta/f_1 g_{2(2)} - g_{2(1)}
\]

so that the secular equation becomes

\[
\Theta(\delta_1, \delta_2, c, \mathcal{R}) = 0
\]

at \( z = 0 \).
Next, to obtain a finite interval of integration, the new coordinate
\[ y = U(z) \]  
and the metric coefficient
\[ x(y) = U'(z) \]  
(together with the new unknowns
\[ h_m(y) = \delta_m(n) \]  
\((m = 2, 3, 4)\) are introduced. Thereby, the last three of Equations (27) become
\[ h'_2 = -h_2h_2' + h_3 + \alpha^2 \]
\[ h'_3 = -h_2h_3' + h_4 + i\alpha e^{\theta}(y - \tau)h_2 - \kappa \]
\[ h'_4 = -h_2h_4' + \alpha^2[h_3 - i\alpha e^{\theta}(y - \tau)] \]
which are to be integrated from \( y = 1 \) to \( y = 0 \). The initial values at \( y = 1 \) for the two fundamental solutions are
\[ h^{(1)}_2 = -\delta \]
\[ h^{(1)}_3 = \Gamma \]
\[ h^{(1)}_4 = 0 \]
\[ h^{(2)}_2 = -\alpha \]
\[ h^{(2)}_3 = 0 \]
\[ h^{(2)}_4 = \alpha \Gamma \]
and the final condition at \( y = 0 \) for a proper value is
\[ \theta = h^{(2)}_2 - h^{(1)}_2 = 0 \]

Now, in place of the original fourth-order linear system over an infinite interval, a third-order non-linear system over a unit interval is involved. Also, whereas the original unknowns \( f_m(z) (m = 1, 2, 3, 4) \) are analytic and thus have only zeros over much of the complex \( z \)-plane, the new unknowns \( h_m(y) (m = 2, 3, 4) \) have poles at points in the complex \( y \)-plane corresponding to those zeros, as evident from Equations (26) and (34). However, while \( f_m(z) \) tend to vary rapidly and strongly in an oscillatory manner, \( h_m(y) \) tend to vary more slowly and weakly in a more monotonic manner except near the poles, where they still vary rather simply. As a net result, in actual calculations, the advantages of the new system substantially outweigh the disadvantages.

In the present method, the integration is performed by expanding the unknowns in power series and then finding the coefficients of the series.
from the differential system*. However, at the conditions of interest, generally at least one singularity of \( h_0(y) \) is close enough to the real interval \( 0 \leq y \leq 1 \) to prevent a suitable representation over that interval by a single expansion in \( y \). Therefore, a sequence of local expansions, adjoined to provide an analytic continuation across the interval, necessarily is employed.

In each local expansion, the local coordinate

\[
x = (y - y_0)/\delta
\]

is used, where \( y_0 \) is a local origin on the real \( y \)-axis** and \( \delta \) is a local expansion radius chosen so that \( 0 \leq x \leq 1 \). Also, the local parameters

\[
y = (x - y_0)/\delta
\]

\[
in \text{a local metrical coefficient}
\]

\[
\lambda(x) = n(y)/\delta
\]

and the local unknowns

\[
\varphi_m(x) = h_m(y)
\]

\((m = 2,3,4)\) are employed. Substituting these quantities, Equations (35) become

\[
\lambda_2' = -\varphi_2' + \varphi_3 + \sigma^2
\]

\[
\lambda_3' = -\varphi_3' + \varphi_4 + \sigma[(x - y)\varphi_3 - \lambda]
\]

\[
\lambda_4' = -\varphi_4' + \sigma^2[\varphi_3 - \sigma(x - y)]
\]

For the first expansion (with \( y_0 = 1 \)), the initial values (at \( x = 0 \)) are

\[
\varphi_2^{(1)} = -\beta \quad \varphi_2^{(2)} = -\alpha
\]

\[
\varphi_3^{(1)} = \gamma \quad \varphi_3^{(2)} = 0
\]

\[
\varphi_4^{(1)} = 0 \quad \varphi_4^{(2)} = \sigma \gamma
\]

*For integrating from \( y = 1 \) (or \( z = \omega \)), this process seems to be preferable to the simpler method of numerical integration, which would entail a troublesome numerical instability at the start of the integration unless excessively many steps were taken there.

**Recent experience suggests that use of a suitable sequence of complex rather than real values of \( y_0 \) could substantially reduce the required number of local expansions.

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whereas, for each subsequent expansion, the initial values are the final values of the preceding expansion. For a proper solution, the final values of the last expansion must satisfy

$$\Theta = \phi_2^{(2)} - \phi_2^{(1)} = 0$$

(at the more important conditions, at least for the Blasius basic flow, the two fundamental solutions tend to become almost identical toward the wall, even in quite improper solutions. Indeed, near the wall, the secular determinant often amounts to such a slight difference in those solutions that it cannot be evaluated at all by just the single-precision arithmetic used in most Fortran programs. In previous techniques, such as the method of numerical integration, this obstacle or its equivalent sometimes is partially overcome by utilizing double-precision arithmetic, which however is cumbersome and costly and still is not adequate in many important situations. Here the obstacle is overcome simply by deducing and applying a supplementary differential system for the secular determinant, which generally can be solved with adequate accuracy by just single-precision arithmetic.

Therefore, only the more gradually varying fundamental solution, found to be \( \phi_2^{(1)} \), is obtained directly from Equations (42) and (43). Then, using that solution in the secular determinant system to be derived, \( \Theta \) rather than \( \phi_2^{(2)} \) is determined.)

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C. SECULAR DETERMINANT

To deduce the secular determinant system, the supplementary local unknowns

$$\eta_m = \varphi_m - \xi_m$$ (2)

are introduced, where \(\eta_m = \varphi_m\) and \(\xi_m = \Theta\). Substituting these in Equations (42) and noting that \(\eta_m\) satisfy those relations, the supplementary local equations

\[
\begin{align*}
\lambda \eta_2' &= -\xi_2 \eta_2 - \xi_2 \eta_2 - \eta_2 \eta_2 + \eta_2 \\
\lambda \eta_3' &= -\xi_3 \eta_3 - \xi_3 \eta_3 - \eta_2 \eta_3 + \eta_4 + \sigma(x - y) \eta_2 \\
\lambda \eta_4' &= -\xi_4 \eta_4 - \xi_4 \eta_4 - \eta_2 \eta_4 - \sigma^2 \eta_3
\end{align*}
\]

(46)

are obtained, in which \(\eta_m\) are to be regarded as known. For the first expansion (with \(p_0 = 1\)), the initial values are

$$\begin{align*}
\eta_2 &= \beta - \sigma \\
\eta_3 &= -\gamma \\
\eta_4 &= \alpha^2
\end{align*}$$

(47)

whereas, for each subsequent expansion, the initial values are the final values of the preceding expansion. For a proper solution, the final value of the last expansion must satisfy

$$\Theta = \eta_4 = 0$$

(48)

As \(\eta_m\) and thus \(\Theta\) become small, Equations (46) approach linear equations with variable coefficients, which vary rather gradually in the critical situations concerned here. Consequently, with due care for a rather abrupt transition from large initial values at \(\gamma = 1\) to very small final values near \(\gamma = 0\), the supplementary system can be solved quite satisfactorily by just single-precision arithmetic*.

*To cover the wide ranges of \(\eta_m\) more easily, a further transformation to unknowns of a more logarithmic nature perhaps would be desirable.
III. EXPANSION SYSTEMS

A. BASIC FLOW

For the integration, the basic flow must be ascertained in a suitable form, which as an illustration now is done for the Blasius basic flow considered in the sample calculations. As explained in Reference 1, this flow is governed by the Blasius differential system

$$2X'' + XX' = 0$$  \hspace{1cm} (49)

and

$$X(0) = X'(0) = 0$$  \hspace{1cm} (50)

$$X'('\infty) = 1$$

where $X$ is a dimensionless stream function of $z$ such that $U = X'$. Here $z$ and $X$ are replaced as variables by $y$ and $\kappa$, yielding the more pertinent system

$$2\kappa' + y = 0$$  \hspace{1cm} (51)

and

$$\kappa'(0) = \kappa(1) = 0$$  \hspace{1cm} (52)

The latter system may be solved by expanding $\kappa$ in the formal series

$$\kappa = \sum_{n=0}^{\infty} \frac{\kappa_3 n y^{3n}}{n!}$$  \hspace{1cm} (53)

where $N$ is a sufficiently large integer and, by a rather elementary process, evaluating the constants $\kappa_n$ ($n = 0, 1, \ldots, N$) from the system. This series is quite simple, and as $N$ increases it converges over the whole interval $0 \leq y \leq 1$. Thus, at least for present purposes, it is preferable to previous representations of the Blasius basic flow, which involve either a more complicated series as in the method of steepest descent (Reference 4), two adjointed series as in the method of Blasius (Reference 2), or tabular data as in the method of numerical integration. However, its rate of convergence is quite slow, due to the presence of a weak singularity at $y = 1$.

Therefore, a more efficient representation in the form of the Padé approximant

$$\kappa = \sum_{n=0}^{M} \frac{\kappa_n y^{3n}}{\kappa_0 + \kappa_3 n y^{3n}}$$  \hspace{1cm} (54)

actually is used, where $\kappa_0 = 1$ and $M$ is a sufficiently large integer. The constants $\kappa_n (n = 0, 1, \ldots, M)$ could be evaluated from $\kappa_n (n = 0, 1, \ldots, 2M + 1)$ by the method of Padé (Reference 7). However, to satisfy the boundary conditions more suitably, they actually are ascertained...
more directly from the differential system, so that the numerator series of the Pade approximant rather than the formal series vanishes exactly at $y = 1$.

The resulting coefficients for $M = 3, 5, \ldots, 13$ are listed in Table 1, and the corresponding approximants are plotted against $y^2$ (which presently is more convenient than $y$) in Figure 1. Also, the numerator zeros in the $y^2$-plane, which will produce singularities in the fundamental solution and secular determinant, and the denominator zeros in that plane are listed in Table 2. These zeros all occur in the real interval $1 \leq y^2 < \infty$ and become more dense as $M$ increases, the approximants in the $y^2$-plane evidently converging to a solution with a branch cut along that interval. Meanwhile, as $M$ increases, the smallest denominator zero approaches 1, and the slope $\kappa'(y)$ at $y = 1$ approaches $-\infty$, which is the correct value. As a net result, even with relatively few terms, the Pade approximant is quite accurate over the interval $0 \leq y < 1$, although its slope $\kappa'(y)$ is finite at $y = 1$ and thus is not entirely typical near that point. Ordinarily, this slight deficiency will not significantly affect the proper solutions, which depend mainly on the nature of the basic flow closer to $y = 0$. In fact, the deficiency prevails over only a tiny range of $y$, which may be regarded as merely the external flow for the boundary layer in the $y$-coordinate in analogy to the boundary layer for the external flow in the $z$-coordinate.

In the sample calculations, just the data for $M = 5$ are used. In other calculations for other boundary layers, the basic flow generally should be representable by similar Pade approximants. In some cases, such as for an asymptotic suction boundary layer, the basic flow can be expressed exactly by a finite Pade approximant.

Therefore, in each local expansion, $\lambda$ is represented as

$$\lambda = \frac{t}{s}$$  \hspace{1cm} (55)

where

$$t = t_{0} e^{\frac{3M}{\nu} \frac{n}{\nu} \frac{m}{\nu}}$$

$$s = s_{0} e^{\frac{3M}{\nu} \frac{n}{\nu} \frac{m}{\nu}}$$ \hspace{1cm} (56)

and $s_{0} = 1$. Substituting from Equations (38) and (40) into Equation (54) and applying the binomial theorem, the local constants are evaluated as

$$t_{m} = t_{0} e^{\frac{3M}{\nu} \frac{n}{\nu} \frac{m}{\nu}}$$

$$s_{m} = s_{0} e^{\frac{3M}{\nu} \frac{n}{\nu} \frac{m}{\nu}}$$ \hspace{1cm} (57)

where

$$t_{m} = \left(t_{0} e^{3M \frac{n}{\nu} \frac{m}{\nu}} \right)^{\frac{m}{\nu}}$$

$$s_{m} = \left(s_{0} e^{3M \frac{n}{\nu} \frac{m}{\nu}} \right)^{\frac{m}{\nu}}$$ \hspace{1cm} (58)
Here \( s_m^n \) are the binomial coefficients, which are obtained recursively as

\[
\begin{align*}
  s_{n-1}^n &= 0 \\
  s_n^n &= 1 \\
  s_m^n &= s_{m-1}^{n-1} + s_{m-1}^{n-1} 
\end{align*}
\]

for \( m = 0, 1, \ldots, 3M \) and \( n = m, \ldots, 3M \). For the first expansion (with \( y_0 = 1 \)), \( t_0 = 0 \) from Equations (52); whereas, for the subsequent expansions, \( t_0 \neq 0 \).
B. FUNDAMENTAL SOLUTION

With the basic flow represented as a rational function, Equations (42) now are expressed in the more convenient form

\[ t_0' = s t_0' \]
\[ t_0' = s t_0' - c t \]
\[ t_0' = s t_0' \]

where

\[ \dot{\theta}_2 = -\phi_2 \dot{\theta}_2 + \varphi_3 + \alpha^2 \]
\[ \dot{\theta}_3 = -\phi_2 \dot{\theta}_3 + \varphi_4 + c(x - \gamma) \phi_2 \]
\[ \dot{\theta}_4 = -\phi_2 \dot{\theta}_4 + \alpha^2 [\varphi_3 - c(x - \gamma)] \]

which will entail only quadratic products of series.

Next, the unknowns are expressed as the formal series

\[ \psi_2 = \sum_{n=0}^{\infty} \frac{p_n}{n!} x^n \]
\[ \psi_3 = \sum_{n=0}^{\infty} \frac{q_n}{n!} x^n \]
\[ \psi_4 = \sum_{n=0}^{\infty} \frac{r_n}{n!} x^n \]

where \( I \) is to be a sufficiently large integer. Substituting these relations and Equations (36) into Equations (60) and (61) and then equating coefficients with the same power of \( x \), the expansion system

\[ \sum_{n-k+1}^{n} k p_k - \psi_0 n-k p_k = 0 \]
\[ \sum_{n-k+1}^{n} k q_k - \psi_0 n-k q_k + c \tau_n = 0 \]
\[ \sum_{n-k+1}^{n} k r_k - \psi_0 n-k r_k = 0 \]

where \( n \geq 0 \) and

\[ p_n + \sum_{n-k}^{n} k p_k - \varphi_n = \alpha^2 \quad (n = 0) \]
\[ = 0 \quad (n \geq 1) \]
Q_n + n\rho_{n-k}Q_k - r_n + \sigma \rho_n
= 0 \quad (n = 0) \quad (cont.) \quad (64)
= \sigma P_{n-1} \quad (n \geq 1)

R_n + \rho_{n-k}R_k - \sigma Q_n
= \sigma \gamma \sigma \nu \quad (n = 0)
= -\sigma^2 \sigma \nu \quad (n = 1)
= 0 \quad (n \geq 2)

is obtained. Both the case \( t_0 = 0 \) for the first expansion and the case \( t_0 \neq 0 \) for each subsequent expansion necessarily are considered.

Later, in solving the expansion system, the summations

\[ a_n = \frac{\sigma}{\gamma} (t_{n-k+1} r_{k+1} - s_{n-k} p_k) \]
\[ b_n = \frac{\sigma}{\gamma} (t_{n-k+1} q_{k+1} - s_{n-k} q_k) \]
\[ c_n = \frac{\sigma}{\gamma} (t_{n-k+1} r_{k+1} - s_{n-k} h_k) \]

where \( z \geq 0 \) and

\[ A_n = 0 \quad (n = 1) \]
\[ = \frac{\sigma}{\gamma} (t_{n-k} r_{n-k} p_k) \quad (n \geq 2) \]
\[ B_n = -\sigma \rho_{n-1} \quad (n = 1) \]
\[ = \frac{\sigma}{\gamma} (t_{n-k} r_{n-k} q_k) \quad (n \geq 2) \]
\[ C_n = \sigma \nu \quad (n = 1) \]
\[ = \frac{\sigma}{\gamma} (t_{n-k} r_{n-k} r_k) \quad (n \geq 2) \]

along with

\[ D_n = \frac{\sigma}{\gamma} (t_{n-k} r_{n-k} p_k - q_n) \]
\[ E_n = \frac{\sigma}{\gamma} (t_{n-k} r_{n-k} q_k - r_n + \sigma \rho_n) \]
\[ F_n = \frac{\sigma}{\gamma} (t_{n-k} r_{n-k} r_k - \sigma \nu \sigma \nu) \]

where \( n \geq 0 \) are used.

For \( t_0 = 0 \) and \( n = 0 \), Equations (63) and (64) degenerate to

-18-
\[ P_0 = 0 \]
\[ Q_0 = 0 \]
\[ R_0 = 0 \]

(68)

and

\[ P_0 P_0 = Q_0 + \alpha^2 \]
\[ P_0 Q_0 = T_0 - \gamma P_0 \]
\[ P_0 R_0 = \frac{\beta}{2} (\eta_0 + \gamma_0) \]

(69)

which have four solutions corresponding to \( F(m) \) (m = 1, 2, 3, 4). Here only the solution corresponding to \( F_1 \) and thus \( \eta_2 \)

\[ P_0 = -\tilde{\beta} \]
\[ Q_0 = \Gamma \]
\[ R_0 = 0 \]

(70)

is needed, which also could be obtained by applying Equations (43) to Equations (62).

For \( T_0 = 0 \) and \( n > 1 \), Equations (63) and (64) become

\[ nT_{fn} - s_n = T_{n-1} - \sigma_n \]
\[ nT_{sn} - s_n = -b_{n-1} - \sigma_n \]
\[ nT_{rn} - s_n = -c_{n-1} \]

(71)

and

\[ P_n + P_{n-1} + P_{n-2} - \eta_n = -A_n \]
\[ Q_n + Q_{n-1} + Q_{n-2} - T_n + \gamma P_n = -B_n \]
\[ R_n + R_{n-1} + R_{n-2} - \alpha^2 \eta_n = -C_n \]

(72)

which necessarily are solved recursively for \( n = 1, 2, \ldots, L \). For each value of \( n \), the right-hand terms are ascertained from preceding data, so that the six equations always constitute an inhomogeneous linear system for the six coefficients \( \eta_n, \gamma_n, \tau_n, P_n, Q_n, R_n \).

The determinant of this system is found to be

\[ \lambda_n = (T_n + 2P_n) \left[ (\tau_n + P_n)^2 - \sigma^2 + P_0^2 - \gamma^2 \right] \]

(73)

-19-
where
\[ T_n = n t_1 / s_0 \] (74)

For the particular solution concerned here, in which \( p_0 = -\beta \), \( \Delta_n \) reduces to
\[ \Delta_n = (T_n - \beta) (T_n - \beta + \alpha) (T_n - \beta - \alpha) \] (75)

and thus has zeros at \( n = 2 \beta s_0 / t_1 \) and \( n = (\beta + \alpha) s_0 / t_1 \). However, in a normal solution, which is the only kind considered here, none of these zeros coincide with \( n = 1, 2, \ldots, \infty \) and the six coefficients always can be determined, the formal series then being valid over their circle of convergence.

In an abnormal solution, to allow a zero at one of the values taken by \( n \), the parameters \( \alpha \) and \( \beta \) and therefore \( \Lambda_2, \Lambda_0, c, \xi \) must have special values depending on \( t_1 \), which itself must be finite because \( \alpha \) and \( \beta \) are finite in the pertinent situations. At these special values, the coefficients for that and the higher values of \( n \) cannot be determined, the formal series then being invalid unless generalized appropriately. For some basic flows, like that in the asymptotic suction boundary layer, \( t_1 \) actually is finite and the abnormal solutions perhaps have a physical significance. However, for the Blasius basic flow, \( t_1 \) really is infinite and the abnormal solutions apparently do not exist. Nevertheless, if \( t_1 \) were too small in the approximate representation of that flow, a zero of \( \Delta_n \) perhaps could occur at one of the values taken by \( n \), causing a misleading result.

Thus, solving Equations (71) and (72) with \( \Delta_n \neq 0 \), the six coefficients are calculated as
\[ p_n = T_n / c_n \]
\[ q_n = -d_n + (T_n + 2 p_0) p_n \]
\[ r_n = -e_n + (T_n + p_0) q_n + (p_0^2 - \beta^2) p_n \] (76)

and
\[ p_n = T_n p_n + s_0^{-1} a_{n-1} \]
\[ q_n = T_n q_n + s_0^{-1} (b_{n-1} + \sigma n) \]
\[ r_n = T_n r_n + s_0^{-1} c_{n-1} \] (77)

where
\[ d_n = -A_n - s_0^{-1} a_{n-1} \]
\[ e_n = -B_n - s_0^{-1} (b_{n-1} + \sigma n) \]
\[ f_n = -C_n - s_0^{-1} c_{n-1} \] (78)
\[
\Gamma_n = \left( (T_n + \rho_0)^2 - \alpha^2 \right) \delta_n + (T_n + \rho_0) \eta_n + \xi_n
\]  
(79)

Together with Equations (68) and (70), the resulting data provide the initial expansion for the fundamental solution.

For each subsequent expansion, in which \( \tau_0 \neq 0 \), the conditions
\[
\begin{align*}
\rho_0 &= \rho_0^* \\
\eta_0 &= \eta_0^*
\end{align*}
\]
(80)

necessarily are observed, where \( \rho_0^*, \eta_0^* \) are the final values at \( x = 1 \) of the preceding expansion. Then, complying with Equations (63) and (64), the remaining coefficients are calculated recursively from
\[
\begin{align*}
\rho_n &= -\rho_n + \alpha^2 \\
\eta_n &= -\eta_n \\
\kappa_n &= -\alpha \rho_n^{n-1} \\
\xi_n &= -\alpha \eta_n^{n} \\
\xi_n &= -\alpha \eta_n^{n-1}
\end{align*}
\]
(81)

and
\[
\begin{align*}
\rho_{n+1} &= -\rho_n/(n + 1) \xi_0 \\
\eta_{n+1} &= -(\rho_n + \alpha \eta_n)/(n + 1) \xi_0 \\
\kappa_{n+1} &= -\alpha \eta_n/(n + 1) \xi_0
\end{align*}
\]
(82)

wherein \( n \geq 0 \).
C. SECULAR DETERMINANT

Using the expression coefficients of both the basic flow and the fundamental solution, the secular determinant is ascertained in a manner similar to that just described. In fact, in the Fortran program, the same subroutines are used for both the fundamental solution and the secular determinant. Hence, just the counterparts of the main equations of the preceding section, identified by the same numbers with an asterisk, are listed here.

Thus, Equations (46) are expressed as

\[ \chi \eta_2 = s \eta_2 \]
\[ \chi \eta_3 = s \eta_3 \]
\[ \chi \eta_4 = s \eta_4 \]

(50*)

where

\[ H_2 = -x_2 \eta_2 - x_2 \eta_2 - \eta_2 \eta_2 + \eta_3 \]
\[ V_3 = -x_2 \eta_3 - x_2 \eta_3 - \eta_2 \eta_3 + \sigma(\lambda - \gamma) \eta_2 \]
\[ H_4 = -x_2 \eta_4 - x_2 \eta_4 - \eta_2 \eta_4 + \sigma^2 \eta_3 \]

(61*)

while the unknowns are represented as

\[ \eta_2 = \sum_{n=0}^{\infty} \eta_{2n} \eta^n \]
\[ \eta_3 = \sum_{n=0}^{\infty} \eta_{2n+1} \eta^n \]
\[ \eta_4 = \sum_{n=0}^{\infty} \eta_{2n+2} \eta^n \]

(62*)

Then, along with Equations (56) and (52), these relations are substituted into Equations (50*) and (61*), yielding the expansion system

\[ \sum_{n=0}^{\infty} \chi_{n-k+1} \eta_{n+k} - \sum_{n=0}^{\infty} \eta_{n-k} \chi_n = 0 \]
\[ \sum_{n=0}^{\infty} \chi_{n-k+1} \eta_{n+k} - \sum_{n=0}^{\infty} \eta_{n-k} \chi_n = 0 \]

(63*)

where \( n \geq 0 \) and

\[ U_n = \chi^n \left( P_{n-k} \eta_k + P_{n-k} \eta_k + u_{n-k} \eta_n \right) - v_n \]
\[ = 0 \quad (n \geq 0) \]
\[ V_n = \chi^n \left( P_{n-k} \eta_k + q \lambda^n \eta_k + u_{n-k} \eta_n \right) - v_n = \sigma v_n \]
\[ = 0 \quad (n \geq 0) \]

(64*)

(contin.)
\[ w_n + \alpha_0^p (p_{n-k}^{u} \kappa_n^k + r_n^{u} u_{n-k}^{w} - u_{n-k}^{w}) - \alpha_0^2 u_n = 0 \] (n > 0)  

As before, the cases \( t = 0 \) and \( t \neq 0 \) for the first and subsequent expansions, respectively, are considered.

Later, the summations
\[
\begin{align*}
\alpha_n^* &= \sum_0^n (t_{n-k}^w) u_{n-k}^w - s_{n-k}^w u_{n-k}^w) \\
\beta_n^* &= \sum_0^n (t_{n-k}^w) v_{n-k}^w - s_{n-k}^w v_{n-k}^w) \\
c_n^* &= \sum_0^n (t_{n-k}^w) w_{n-k}^w - s_{n-k}^w w_{n-k}^w)
\end{align*}
\]

where \( n \geq 0 \) and
\[
\begin{align*}
A_n^* &= 0 & (n = 1) \\
B_n^* &= \sum_1^{n-1} (p_{n-k}^{u} + p_{n-k}^{w} + u_{n-k}^{w}) & (n \geq 2) \\
C_n^* &= -\sum_{n-1}^{n} (q_{n-k}^{u} + q_{n-k}^{w} + u_{n-k}^{w}) - \sigma u_{n-1} & (n \geq 2) \\
D_n^* &= 0 & (n = 1) \\
E_n^* &= \sum_1^{n-1} (p_{n-k}^{u} + q_{n-k}^{w} + u_{n-k}^{w}) & (n \geq 2)
\end{align*}
\]

along with
\[
\begin{align*}
\beta_0^* &= \sum_0^{n} (p_{n-k}^{u} + p_{n-k}^{w} + u_{n-k}^{w}) - v_n \\
\epsilon_0^* &= \sum_0^{n} (q_{n-k}^{u} + q_{n-k}^{w} + u_{n-k}^{w}) - u_n + \sigma u_n \\
\delta_n^* &= \sum_0^{n} (t_{n-k}^w + r_n^{u} u_{n-k}^{w} + u_{n-k}^{w}) - \alpha^2 u_n
\end{align*}
\]

where \( n \geq 0 \) are used.

For \( t = 0 \), the solution at \( n = 0 \) is
\[
\begin{align*}
V_0 &= 0 \\
\dot{V}_0 &= 0 \\
\ddot{V}_0 &= 0
\end{align*}
\]

and
\[
\begin{align*}
u_0 &= \beta - \sigma \\
v_0 &= -\Gamma \\
v_0 &= \sigma g
\end{align*}
\]

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whereas the recursive system at n = 1, 2, ..., L is

\[ n t \nu_n - s_0 \nu_n = -s_{n-1} \]

\[ n \nu_n - s_0 \nu_n = -b_{n-1} \]

\[ n t \nu_n - s_0 \nu_n = -c_{n-1} \]

and

\[ u_0 + p_0 u_0 + p_0 u_0 + u_0 v_0 \]

\[ + p_0 u_0 + p_0 u_0 + u_0 v_0 - v_n = -A_n \]

\[ v_n + p_0 v_n + q_0 v_0 + u_0 v_n - w_n = -B_n \]

\[ u_0 + p_0 u_0 + p_0 u_0 + u_0 v_0 \]

\[ + p_0 u_0 + p_0 u_0 + u_0 v_0 - s_0 v_n = -C_n \]

which is solved for \( u_n, v_n, w_n, u_n, v_n, w_n \).

Here the determinant of the system is

\[ \Delta_n = (T_n + 2p_0 + 2u_0) [(T_n + p_0 + u_0)^2 - \sigma^2 + (p_0 + u_0)^2 - s^2] \]

which for the solution concerned reduces to

\[ \Delta_n = (T_n + 2p)(T_n - \alpha - \beta)(T_n - \alpha + \beta) \]

and has zeros at \( n = 2p_0/t_1 \) and \( n = (\alpha \pm \beta)/t_1 \). Again, only a normal solution avoiding such zeros is considered, the formal series then being valid over their circle of convergence. However, in singular circumstances like those discussed before, an abnormal solution possibly could occur.

Accordingly, for the first expansion, the remaining coefficients (\( a_n, n \geq 1 \)) are calculated as

\[ u_n = T_n u_0 \]

\[ v_n = -d_n + (T_n + 2p_0 + 2u_0)u_n \]

\[ w_n = -c_n + (T_n + p_0 + u_0)v_n + [(p_0 + u_0)^2 - \sigma^2]v_n \]

and

\[ u_n = T_n u_0 \]

\[ v_n = -d_{n-1} + \frac{1}{t_0} u_{n-1} \]

\[ w_n = T_n v_0 + \frac{1}{t_0} v_{n-1} \]

\[ -24- \]
where
\[ s^*_n = -a^*_n - s_{n-1}^* - u_0 p_n - v_0 p_n \]
\[ e^*_n = -b^*_n - s_{n-1}^* - v_0 p_n - u_0 q_n \]
\[ r^*_n = -c^*_n - s_{n-1}^* - v_0 p_n - u_0 r_n \]

and
\[ I^*_n = [\tau^*_n + p_0 - u_0]^2 \cdot e^*_n + (\tau^*_n + p_0 + u_0) e^*_n + r^*_n \]

For each subsequent expansion, wherein \( \tau_0 \neq 0 \), the conditions
\[ v_0 = u_0^* \]
\[ v_0 = u_0^* \]
\[ u_0 = u_0^* \]

are observed, where \( u_0^* \), \( v_0^* \), \( w_0^* \) are the final values at \( x = 1 \) of the preceding expansion, and the remaining coefficients are calculated from
\[ u_n = -u^*_n \quad (n \geq 0) \]
\[ v_n = -v^*_n \quad (n = 0) \]
\[ w_n = -w^*_n \quad (n \geq 1) \]

and
\[ u_{n+1} = -a_n^*/(n + 1)\tau_0 \]
\[ v_{n+1} = -b_n^*/(n + 1)\tau_0 \]
\[ w_{n+1} = -c_n^*/(n + 1)\tau_0 \]

wherin \( n \geq 0 \). For a proper solution, the final value \( u^*_n \) at \( x = 1 \) of the last expansion, which is the value of \( \theta \) at \( y = z > 0 \), must vanish.
IV. AUXILIARY RELATIONS

A. RATIONAL EXPANSIONS

In each expansion, whereas the origin $y_0$ is predetermined, the radius $\delta$ is somewhat flexible and can be chosen best only after the expansion coefficients for a trial value have been found. Therefore, in the Fortran program, the expansion coefficients first are computed from the preceding relations for a tentative value of $\delta$, which is selected so as to avoid troublesome truncation and overflow or underflow errors and to keep the expansion interval within the applicable range of $y$. Then, a convergence test is applied to these coefficients to find a preferred value of $\delta$, which would provide a moderate rate of convergence at $x = 1$. Finally, the expansion coefficients either are left unchanged or are scaled down by an elementary process to the preferred value, according to whether the tentative value is smaller or larger than the preferred value. Thereby, the final formal series always converge at least moderately over $0 \leq x < 1$.

However, to accelerate the convergence and thus minimize the number of terms required for a given precision, the formal series subsequently are converted into Padé approximants. Thus, representing each final formal series as

$$ f = \sum_{n=0}^{\infty} a_n x^n $$

where $a_n (n = 0, 1, \ldots, L)$ are known constants, the rational function

$$ f = \frac{\sum_{n=0}^{L} b_n x^n}{\sum_{n=0}^{L} c_n x^n} $$

always is constructed. Here $L = L - 1 + \mu$ and $c_0 = 1$, whereas $\mu$ is an arbitrary positive integer and $p_n (n = 0, 1, \ldots, \mu)$, $q_n (n = 1, 2, \ldots, \mu)$, and $c_0$ are constants determined from $a_n (n = 0, 1, \ldots, L)$. The last numerator term is the leading term of the truncation error and is omitted in the actual evaluation of the approximant; the coefficient $a_0$, serving only as an error index.

To determine the rational coefficients, Equations (83) and (84) are multiplied by the denominator of Equation (84), and the resulting coefficients with the same power of $x$ are equated. This process yields

$$ p_n = \sum_{k=0}^{\mu} \alpha_{n-k} b_k $$

for $0 \leq n \leq \mu$, where $\alpha_0 = \min (n, \mu)$, and

$$ q_{n-k} c_k = -p_n a_0 $$

for $\mu + 1 \leq n \leq \mu + \mu + \nu$ together with

$$ a_0 = p_0 n^{-k} k $$

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for \( n = \mu + \nu + 1 = L \). Equations (66) constitute an inhomogeneous linear system for the unknowns \( q_0, q_1, \ldots, q_L \), which is solved by the elimination method (Reference 8) assuming that the determinant of the system does not vanish. Then, using that solution, \( p_{n} (n = 0, 1, \ldots, \mu) \) and \( \epsilon_0 \) are computed directly from Equations (85) and (87), respectively.

In the complex \( z \)-plane, as \( L \) increases, the formal series converges only within the circle centered at the origin and extended to the nearest singularity. Meanwhile, as \( \mu \) and \( \nu \) increase, the rational function (even though constructed from the formal series) converges over a much larger region, which apparently includes the whole finite plane except for infinitesimal regions around the poles and around the rays emanating from the other singularities to infinity. Furthermore, the superior convergence of the rational function persists into the circle of convergence of the formal series. As a net result, in the present computations, a given precision can be attained more economically by obtaining and using the rational function, despite the extra relations entailed. In fact, in place of a sequence of local expansions, just a single rational expansion at \( y = 1 \) would suffice theoretically, since it would converge at \( y = 0 \). However, in most cases of interest, the rounding and other errors then would be too troublesome, the procedure followed here being a practical compromise.

In the present Fortran program, to limit the calculations and errors, the restrictions \( L = \min (L_0, \mu) \) and \( \nu = \min (\mu + 1) \) are imposed. Here, \( L_0 \) is the maximum degree of the formal series, which is chosen from \( 3 \leq L_0 \leq 29 \), and \( \mu \) is the minimum degree of that series providing moderate convergence, which is calculated along with the expansion coefficients. In the sample calculations, the value \( L_0 = 24 \) generally was used, and the condition \( L = L_0 \) commonly occurred. As a result, the number of local expansions varied from about 11 to over 30, depending on the values of \( A_1, A_2, \epsilon_0 \), and \( \mu \).
B. ROOT EXTRACTION

As indicated earlier, when $A_2,c,R$ are specified, the proper values of $A_1$ are just the zeros of $\Theta$ at $n = 0$, to be denoted as $\Theta^*(A_j)$. However, these zeros cannot be expressed explicitly in a tractable way, owing to the intricacy of $\Theta^*(A_j)$. Therefore, to ascertain a proper value, first a local polynomial approximation of $\Theta^*(A_j)$ is established. Then, the pertinent root of that polynomial is located and used as an approximation of the proper value.

Thus, for each proper value, a small set of adjacent values of $A_1$, denoted as $A_{1}^{(j)}$ $(j = 0, 1, \ldots, n - 1)$ where $n$ is the number of such values, is selected. To minimize the computations for a given precision, these values are distributed equiangularly around a circle in the complex $A_1$-plane chosen so that the center $A_1$ is as close to the proper value as possible and the radius $R_1$ is as small as tolerable. Thus, the adjacent values are

$$A_{1}^{(j)} = \bar{A}_1 + R_1 E_j$$  \hspace{1cm} (88)

where each $E_j$ is a unit circle value

$$E_j = \exp(\text{i}2\pi j/n)$$  \hspace{1cm} (89)

of another complex variable $E$. Next, the quantities $\Theta_j = \Theta^*(A_{1}^{(j)})$ are calculated by the preceding method and used to construct the interpolation polynomial

$$\Theta_k = \sum_{j=0}^{n-1} c_k E_j^k$$  \hspace{1cm} (90)

where $c_k$ $(k = 0, 1, \ldots, n - 1)$ are constants such that $\Theta_k = \Theta_j$ at $E = E_j$. In this construction, the identity

$$\sum_{j=0}^{n-1} E_j^k = n$$ \hspace{1cm} (4 = 0) \hspace{1cm} (91)

$$= 0$$ \hspace{1cm} (4 \neq 0)

for $k = 0, 1, \ldots, c(n-1)$ is employed, yielding

$$\sum_{j=0}^{n-1} \Theta_j E_j^k = \sum_{j=0}^{n-1} c_k E_j^k \sum_{j=0}^{n-1} E_j^k = c_k n$$ \hspace{1cm} (92)

whereupon

$$c_k = n^{-1} \sum_{j=0}^{n-1} \Theta_j E_j^k$$  \hspace{1cm} (92)
Except as limited by the precision of the arithmetic, the resulting polynomial approximates the first \( n \) terms of a complex Taylor series about \( z = 0 \) and thus, as \( n \) increases, converges in the circle centered at that point and extending to the nearest singularity (generally a pole). This particular interpolation scheme, perhaps somewhat new, is a complex-variable counterpart of the highly-efficient real-variable Chebyshev interpolation scheme. At the larger values of \( n \), to further improve the convergence and thereby the accuracy of the resulting root (particularly near a pole), the interpolation polynomial is converted into the Padé approximant

\[
\Theta(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_n}{z^n}
\]

where \( c_0 = 1 \) and \( v \) is an arbitrary integer, while \( c_k (k = 0, 1, \ldots, n-1) \) and \( \delta_k (k = 1, 2, \ldots, v) \) are constants determined from \( \delta_k (k = 0, 1, \ldots, n-1) \) by the method described in Section (F-1). Finally, the pertinent root \( \delta_k \) of

\[
\sqrt[n-1]{\delta_k} \epsilon_k = 0
\]

or

\[
\sqrt[n-1]{\delta_k} \epsilon_k = 0
\]

as applicable is located by Newton's method (for simple roots) and, if within the range of the adjacent values, is used in the approximation

\[
A^*_1 = I_1 + \frac{\epsilon_k}{\delta_k}
\]

for the desired proper value. Subsequently, as necessary, the process is repeated until sufficient accuracy is attained. With due care for truncation errors, particularly those inherent in each \( \Theta \), all calculations generally can be performed satisfactorily with just single-precision arithmetic.

In the present Fortran program, the root is extracted from the polynomial if \( 2 \leq n \leq 4 \) and from the rational function if \( 5 \leq n \leq 16 \) (a maximum of 16 adjacent values being allowed). In the latter case, to further improve the accuracy, a second extraction from a polynomial centered at the root from the rational function is included (with \( 2 \leq n \leq 8 \)). In both cases, as a check, the value of \( \Theta^*(A^*_1) \) at the final root, which in an exact calculation would vanish, also is obtained. When this check value significantly exceeds the inherent computational errors, the program is rerun, using the final root of the preceding run as the initial center. In the sample calculations, for example, the value \( n = 6 \) was chosen (with \( n = 3 \) in the second extraction). For most runs, the interpolation radii were taken as rather small fractions of the proper value magnitude.

Ordinarily, just the principal proper value, which has the algebraically smallest imaginary part, is sought. However, at least in principle, several higher proper values also could be ascertained by the foregoing procedure. In general, the significance of the higher proper values has not yet

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been adequately assessed, especially in regard to the resonance theory of
transition. Some instructive explorations of such values for relatively
simple basic flows are described in References 9 thru 11.
As in most other methods of solving the Orr-Sommerfeld system, the calculations entailed here are too extensive to perform manually in a practical way, even for a single proper solution. Therefore, a Fortran program (References 12 and 13) for conducting them on a digital computer (the IBM 7090 data processing system at the Naval Division) necessarily was developed.

At present, the whole program includes one main program (MPA) using eleven subroutines (SR8 thru SR1) and another main program (MPM) using the same subroutines plus two additional subroutines (SRW and SR0). The titles, cosmic notation, and source statements of these routines are listed in Appendix I. This information together with the preceding analysis and the following remarks indicates the general nature of the program. Complete details would be too lengthy to describe here.

In its present form, the program includes a few vestiges from earlier programs that were tried without adequate success. Also, it costs labels from output data, which therefore must be identified from the listed notation and output statements. Moreover, various generalizations, such as to include the useful bypassed equations and to cover the adjoint as well as the actual Orr-Sommerfeld system, are underway or contemplated. Consequently, the program is somewhat tentative and later may be refined and revised.

A. ADJOINTED POWER SERIES SOLUTION

The main purpose of program MPA (adjoint power series solution) is to obtain the expansion coefficients of the fundamental solution and secular determinant for specified values of $A_1, A_2, c, R$. When $A_1$ has a proper value, these coefficients readily yield the corresponding proper function. Another purpose is to provide the values of $\Theta$ at $z = 0$ for sets of values of $A_1, A_2, c, R$ from which the proper values can be estimated in a preliminary manner.

The principal subroutine is SRG, which is performed once for each local expansion. It calculates the formal expansion coefficients of both the fundamental solution and the secular determinant by the relations of Sections (E-2) and (E-3), using SRH thru SRJ for the summations involved. Then, from these coefficients, it calculates the corresponding rational expansion coefficients by the relations of Section (F-1), using SRK for this purpose. In turn, the last routine employs SRK to solve the complex linear system thus encountered.

The local expansions are adjointed by SR4, which also provides the local basic flow coefficients and binomial coefficients supplied by SR8 along with parameters computed in SRD. The secular determinant at the wall is calculated by SRG.

*Further details can be supplied upon request.

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3. PROPER VALUE LOCATION

The main purpose of program MPM (proper value location) is to find the proper values of \( \lambda_1 \) when \( \lambda_2, \lambda_3, \lambda \) are specified, which are needed to evaluate the resonance growth functions of Reference 1. Another purpose is to find the proper values of \( \lambda \) when \( \lambda_1, \lambda_2, \lambda_3 \) are specified, which are needed to evaluate the resonance coefficients of Reference 1.

Here, the primary subroutine is SRC, which employs SRD thru SRL to provide the value of \( \Theta \) at \( z = 0 \) for each adjacent value of \( \lambda_1 \) (or \( \lambda \)). From these data, the proper value is extracted by the relations of Section (F-2), using SRH for the interpolation polynomial coefficients and SRK for the rational function coefficients together with SRC for the zero location.
VI. SAMPLE CALCULATIONS

To demonstrate the method and provide basic data for research on transition, some proper values and functions for the Blasius basic flow were calculated. The principal mode, for which Im(A_j) has the algebraically smallest value, and two-dimensional perturbations, for which A_2 = A_1 = 0, were considered. To otherwise cover the main regions of most actual transitions in an efficient way, the values of c and R were selected as

\[ c_j = -0.005 \exp[\log(20) \sin^2(j\pi/2n)] \]

\[ R_k = 125 \exp[\log(20) \sin^2(k\pi/2n)] \]

(97)

where j, k = 0, 1, ..., n and n = 6. These 69 points encompass the ranges of Equations (25) and are distributed so as to allow Chebychev interpolations in the logarithms of c and R.

Each point required an automatic computer time of about 0.1 hour or more and thus was rather expensive, which emphasizes the advisability of economizing throughout the program and calculation. In previous computational schemes, some of the points would have required double- or triple-precision arithmetic, which perhaps would have increased the time for those points about four or nine times, respectively, or more.

The accuracy of the resulting data varies somewhat, being least where c and R are greatest. Near c = -100 and R = 2500, which is the most critical region, the error in the proper values apparently is of the order of 0.001 percent of the absolute value. The accuracy of the proper functions should equal that of the proper values, since their greatest error tends to occur near the wall where the proper values are determined. However, in previous computational schemes, the proper functions evidently can be less accurate than the proper values.

A. PROPER VALUES

The proper value of A_j as a function of c and R is listed in Table 3 and plotted in Figure 2. Here A_j is complex while c and R are real, unlike conventional linear stability data in which c is complex while A_j and R are real. Thus, these data pertain to spacewise modulations of Fourier components of the motion, which are needed in most applications, whereas the conventional data represent time-wise modulations of those components. In previous analyses, the spacewise variations usually have been merely estimated from the time-wise variations, which sometimes can be done without excessive error (Reference 16). However, at least as ordinarily performed, such estimates become poorer as the phase velocity of the Fourier component decreases and in fact are invalid where that velocity vanishes, which happens when stationary waves occur.

For the particular basic flow and conditions considered here, the proper values near and inside the neutral curve, on which Im(A_j) = 0, agree reasonably well with previous double-precision calculations (Reference 13).
However, the proper functions themselves may differ more substantially, especially at the larger values of c and k, the present data presumably being the more accurate. The proper values further outside the neutral curve cannot be compared, because they have been omitted from the previous calculations. Nevertheless, such values are somewhat important, since some resonance growth can occur outside the neutral curve, contrary to linear stability theory which predicts only damping in that region.

B. PROPER FUNCTIONS

The fundamental solution \( f_2 \) and secular determinant \( \eta_2 \) composing each proper function, for the values of Equation (97) with \( j, k = 0, n/2, n \) only, are represented as functions of \( y \) in Figure 3. For convenience, in place of \( \eta_2 \), the more tractable parameter \( \log(\eta_2) \) is plotted.

Clearly, as anticipated, \( f_2 \) generally is remarkably smooth, whereas the original variable

\[
\xi_1 = \exp \left[ \int_0^y f_2 \, dy \right]
\]

(98)

varies greatly in an oscillatory manner, particularly at the higher values of \( c \) and \( k \). Contrarily, \( \eta_2 \) itself varies rather strongly, although the parameter representing this quantity in Figure 3 is almost as smooth as \( f_2 \). These relatively simple and mild trends facilitate the calculations and thus help to justify the elaboration of the method. They also suggest that useful asymptotic approximations, differing from those of the familiar method of asymptotic expansions (References 2 thru 4), perhaps could be established by further investigation.
Insofar as observed, the Fortran program described here should be satisfactory as a basis for implementing the resonance theory of transition, and it incidently should be valuable for extending the linear theory of instability. To fully exploit the possibilities, though, various generalizations are desirable. The most urgent is the incorporation of the pertinent bypassed equations and the adjoint Orr-Sommerfeld system, so as to evaluate the resonance coefficient of Reference 1. Others include extensions to three-dimensional curvilinear coordinates, for handling curvilinear phenomena like Soretier and crossflow vortices, and to compressible and real fluids, for investigating supersonic and hypersonic transitions. In general, the present technique should be somewhat more economical and dependable than previous techniques, and it perhaps could be improved significantly by further development.

Meanwhile, continuation of the present calculations to broader conditions and additional basic flows would be appropriate. In particular, some higher modes and the three-dimensional perturbations should be covered. Indeed, as mentioned in Reference 1, the proper solutions for \( \Re(b_1) = \lambda_1 = 0 \), which represent streamwise (not crossflow) vortices, may provide insight into the nature of turbulent wedges. Also, systematic data for special basic flows such as the Hartree flows with suction and viscoelastic walls would be valuable for general reference. In this connection, the proper values for the asymptotic suction profile, as obtained by a predecessor of the present method, are included in Figure 4. For these calculations, just single-precision arithmetic (8 decimal places) was used, but the accuracy exceeds that attainable from quadruple-precision arithmetic (12 decimal places) in previous schemes. Altogether, considering the growing importance of transition in technology and the wide range of conditions encountered, such calculations could be continued in a worthwhile way rather indefinitely.

Also, separate programs are needed for actually evaluating the resonance coefficients and growth functions of Reference 1 from the proper solutions. To indicate the significance of such growth functions, the downstream modulations of a typical Fourier component of the motion according to the resonance and linear theories, as estimated from Figure 7 in an approximate way, are compared in Figure 5. In this particular case, the two theories diverge greatly near the lower branch of the neutral curve. In other cases, they would differ substantially in other ways, yielding greatly different whole motions.
REFERENCES


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**Table 3: Power Values for Blasius' Basic Flow**

\( A_1 + A_2 + \sigma = 0 \)
FIGURE 2  PROPER VALUES FOR BLASIUS BOUNDARY LAYER
FIGURE 3  PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(A) FUNDAMENTAL SOLUTION (REAL PART)
FIGURE 3 PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(A) FUNDAMENTAL SOLUTION (IMAGINARY PART)
FIGURE 3  PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(B) SECULAR DETERMINANT (REAL PART)
FIGURE 3  PROPER FUNCTIONS FOR BLASIUS BASIC FLOW
(B) SECULAR DETERMINANT (IMAGINARY PART)
Figure 5: Typical Growth Functions in Blasius Boundary Layer
### A. TITLES

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B. COMMON NOTATION

Subscripts used below:  
I = expansion number  
J = j + 1  
K = k + 1  
L = root number

\[ B(1) = A_1 \]
\[ B(2) = A_2 \]
\[ B(3) = c \]
\[ B(4) = R \]
\[ B(5) = \alpha^2 \]
\[ B(6) = \alpha \]
\[ B(7) = \alpha^{-1} \]
\[ B(8) = a \]
\[ B(9) = b \]
\[ B(10) = c \]
\[ B(11) = iR \]
\[ B(12) = \text{Im}R \]
\[ B(13) = \text{Im}Re \]
\[ B(14) = \Gamma \]
\[ B(15) = \beta^2 \]
\[ B(16) = \beta \]
\[ B(17) = \alpha^2 \Gamma \]
\[ B(18) = \text{vacant} \]
\[ B(19) = \text{vacant} \]
\[ B(20) = \text{vacant} \]
C (1,1) = \gamma \\
C (1,2) = \sigma \\
C (1,3) = \gamma \sigma \\
C (1,4) = \sigma^2 \sigma \\
C (1,5) = \sigma^2 \gamma \sigma \\
C (1,6) = \text{vacant} \\
C (1,7) = \text{vacant} \\
C (1,8) = \text{vacant} \\
D (1) = h_2^{(1)} (0) \\
D (2) = h_2^{(2)} (0) \\
D (3) = \Theta^* \\
D (4) = \text{vacant} \\
D (5) = \text{vacant} \\
D (6) = \text{vacant} \\
D (7) = \text{vacant} \\
D (8) = \text{vacant} \\
D (9) = (S_2, \text{BH})^* \\
G (J,K) = g_j^k \\
H (I + 1,1) = \eta_2 (1) \\
H (I + 1,2) = \eta_3 (1) \\
H (I + 1,3) = \eta_4 (1) \\
H (I + 1,4) = \eta_2 (1) \\

* S_2 = \gamma_0 + \delta \text{ of last expansion} \\
\text{HM} = \text{maximum error index of all expansions}
H (I + 1,3) = \eta_3 (1)

h (I + 1,6) = \eta_4 (1)

H (I + 1,7) = \text{error index for } \eta_2

H (I + 1,8) = \text{error index for } \eta_3

H (I + 1,9) = \text{error index for } \eta_4

H (I + 1,10) = \text{error index for } \eta_2

H (I + 1,11) = \text{error index for } \eta_3

H (I + 1,12) = \text{error index for } \eta_4

K (1) = \text{adjacent values (initial interpolation)}

K (2) = \text{adjacent values (final interpolation)}

K (3) = \text{unknown (streamwise frequency or Reynolds number)}

K (4) = \text{roots sought (each run)}

K (5) = \text{iterations (root extraction)}

K (6) = \text{printing (maximum or minimum)}

K (7) = \text{interpolation (initial or final)}

K (8) = \text{terms (in polynomial or Padé numerator)}

K (9) = \text{roots calculated (each interpolation)}

L (1) = \text{expansion number}

L (2) = \text{expansion type (initial or subsequent)}

L (3) = n (term degree)

L (4) = n + 1 (term number)

L (5) = n + 2

L (6) = \text{rational functions (1 to 6)}
\( M (1) \) = numerator terms (basic flow)
\( M (2) \) = denominator terms (basic flow)
\( M (3) \) = expansions (per solution)
\( M (4) \) = maximum terms (per expansion)
\( M (5) \) = printing (maximum, medium, or minimum)
\( M (6) \) = vacant
\( M (7) \) = vacant
\( M (8) \) = vacant
\( M (9) \) = vacant
\( M (10) \) = maximum of \( M (1) \) and \( M (2) \)

\( N (1,1) \) = expansion type (initial or subsequent)
\( N (1,2) \) = total terms
\( N (1,3) \) = numerator terms
\( N (1,4) \) = denominator terms (assigned)

\( \Theta (1,K) = \xi_k^0 \)
\( \Theta (2,K) = \eta_k^0 \)

\( P (1,1,K) = p_k \)
\( P (1,2,K) = q_k \)
\( P (1,3,K) = r_k \)
\( P (1,4,K) = u_k \)
\( P (1,5,K) = v_k \)
\( P (1,6,K) = w_k \)

-52-
\[ Q(1, K) = F_k \]
\[ Q(2, K) = G_k \]
\[ Q(3, K) = R_k \]
\[ Q(4, K) = U_k \]
\[ Q(5, K) = V_k \]
\[ Q(6, K) = W_k \]

\[ R(1,1,K) = r_k \]
\[ R(1,2,K) = s_k \]

\[ S(I,1) = \delta \text{ (tentative value)} \]
\[ S(I,2) = y_o \]
\[ S(I,3) = \lambda_o \]
\[ S(I,4) = \delta \text{ (final value)} \]

\[ U(1) = a_{n-1} \quad \text{or} \quad a_n \]
\[ U(2) = b_{n-1} + ct_n \quad \text{or} \quad b_n + ct_n \]
\[ U(3) = c_{n-1} \quad \text{or} \quad c_n \]
\[ U(4) = a^*_{n-1} \quad \text{or} \quad a^*_n \]
\[ U(5) = b^*_{n-1} \quad \text{or} \quad b^*_n \]
\[ U(6) = c^*_{n-1} \quad \text{or} \quad c^*_n \]

\[ V(1) = d_n \quad \text{or} \quad D_n \]
\[ V(2) = e_n \quad \text{or} \quad E_n \]
\[ V(3) = f_n \quad \text{or} \quad F_n \]
\[ V(4) = \phi_n \] or \[ D_n \]

\[ V(5) = \psi_n \] or \[ E_n \]

\[ V(6) = \xi_n \] or \[ F_n \]

\[ X(1, J) = Y_j / n \] (for initial interpolation)

\[ X(2, J) = \bar{Y}_j / n \] (for final interpolation)

\[ Y(1, J) = \Theta_j \]

\[ Y(2, J) = a_j \]

\[ Y(3, J) = \text{polynomial coefficients or Fade coefficients (numerator and denominator)} \]

\[ Z(1, L) = \text{initial root (unit-circle or actual)} \]

\[ Z(2, L) = \text{final root (unit-circle or actual)} \]

\[ Z(3, L) = \text{error in final root} \]
C. SOURCE STATEMENTS
MAIN PROGRAM MPA

ADJOINED POWER SERIES SOLUTION

USES SRL THRU SRL

USES 10 DATA CARDS FOR BASIC FLOW*****

2 * * M(1)***29 NUMERATOR TERMS (BASIC FLOW)
2 * * M(2)***29 DENOMINATOR TERMS (BASIC FLOW)

PLUS 2 DATA CARDS FOR CONTROL*****

4 * * M(4)***30 MAXIMUM TERMS (PER EXPANSION)
1 * * M(5)***3 PRINTING (MAX, MED, MIN)

PLUS 6 DATA CARDS FOR PARAMETERS*****

A(1,J) STREAMWISE FREQUENCY (REAL PART)
A(2,J) STREAMWISE FREQUENCY (IMAG PART)
A(3,J) CROSSWISE FREQUENCY (REAL PART)
A(4,J) CROSSWISE FREQUENCY (REAL PART)
A(5,J) REYNOLDS NUMBER (REAL PART)
A(6,J) REYNOLDS NUMBER (IMAG PART)

DIMENSION L(6,J=10)***N(30+5)

1 R(30+2+29)***G(29+29)+D(2+29)

2 A(6,5)

1 DIMENSION H(31,12)***Q(30+5)

1 B(20)

COMMON L*M*N*M*P*Q*R+S+S+O+B

CALL SRL

10 READ INPUT TAPE 5+30* (L(1)*A(1,J),J=1,5) ,L=1,6)

30 FORMAT (112,5E12,8)

L1=L(1)
L2=L(2)
L3=L(3)
L4=L(4)
L5=L(5)
L6=L(6)
L(6)=L
B1Z2=0.
B(23)=0
DO 150 I1=1,L1
B(1)=A(1,11)
DO 140 I2=1,L2
B(2)=A(2,12)
DO 130 I3=1,L3
B(2)=A(3,13)
DO 120 I4=1,L4
B(3)=A(4,14)
DO 110 I5=1,L5
B(4)=A(5,15)
DO 100 I6=1,L6
B(24)=A(6,16)
CALL SRC
100 CONTINUE
110 CONTINUE
120 CONTINUE
130 CONTINUE
140 CONTINUE
150 CONTINUE
GO TO 10
END(10) 930T4000
SUBROUTINE SRB
C BASIC FLOW AND CONSTANTS
C 0(2,1) MUST EQUAL 1
C
DIMENSION L(6), M(10), N(30+4)
1 R(30+2,29) = S(30) + G(29+29) * O(2,29)
1 DIMENSION M(31,12), P(50,50), Q(6,48)
COMMON Lx,Mx,Nx,Px,Rx,Sx,Gx
READ INPUT TAPE 5,10, M(1) = (0(1,K), K = 1, 29)
READ INPUT TAPE 5,10, M(2) = (0(2,K), K = 1, 29)
READ INPUT TAPE 5,10, M(4)
READ INPUT TAPE 5,10, M(5)
10 FORMAT (1L2+9E12,8/16E12,8)
WRITE OUTPUT TAPE 6,20
20 FORMAT (1H1)
28 WRITE OUTPUT TAPE 6,30, M(1) = (0(1,K), K = 1, 29)
WRITE OUTPUT TAPE 6,30, M(2) = (0(2,K), K = 1, 29)
30 FORMAT (1Hus,19+9E20,8/1Hus,9+9E20,8)
M(1) = XMAXOF(M(1)+2)
M(2) = XMAXOF(M(2)+2)
M(4) = XMAXOF(M(4)+4)
M(1,1) = 1
DO 40 I = 2, 30
40 M(I,1) = 2
DO 60 J = 7, 12
160 M(1,J) = (0+0)
M(12) = XMAXOF(M(1), M(2))
M(10) = M(12)
G(J,1) = 1
70 DO 100 J = 2, M(12)
G(J,J1) = 1
J1 = J - 1
G(J,J1) = 0
80 DO 90 I = 1, 2
90 G(I,J) = G(I,J1) + G(I-1,J1)
100 CONTINUE
RETURN
SUBROUTINE SRC
C SECULAR DETERMINANT
C USES SRD THRU SRL
C
DIMENSION L(6), M(10), N(30+4),
1 R(30+2,29), S(30+4), G(29,29), O(2,29)
I
DIMENSION H(31,12), P(30+6,30), Q(6,48),
1 B(20)+C(30+B)+O(9)
COMMON L*M*N*H*P*Q*R*S*G*O*B*C*D
CALL SRD
I
H(1,1)=B(16)
I
H(1,2)=B(14)
I
H(1,3)=(0+0x)
I
H(1,4)=B(16)-B(6)
I
H(1,5)=B(14)
I
H(1,6)=B(17)
CALL SRE
M5=M(5)
10 GO TO (20+30+50)+M5
20 CONTINUE
30 WRITE OUTPUT TAPE 6+40+, ID(I), DI(1+9), II(1+3)
40 FORMAT (2HO+19+8+5E2O+8)
50 RETURN
FREQUENCY 10(1+10+100)
END (0) 990TC000
990TC010
990TC020
990TC030
990TC040
990TC050
990TC060
990TC070
990TC080
990TC090
990TC100
990TC110
990TC120
990TC130
990TC140
990TC150
990TC160
990TC170
990TC180
990TC190
990TC200
990TC210
990TC220
990TC230
990TC240
990TC250
990TC260
990TC270
990TC280
990TC290
990TC300
990TC310
SUBROUTINE SRD
C PARAMETERS
C
DIMENSION L(6)*M(10)*N(30+4)
1 K(30+2*29)+S(30+4)*G(29+29)+O(2+29)
1 DIMENSION H(31+12)+P(30+6+30)+Q(6+48)
1 COMMON L*W*N*H*P*G*S*O+B
1 B(5)*B(1)*B(2)**2+B(2)**2
1 B(6)=SQRTF(B(5))
1 B(7)*I*0/BI/B(6)
1 B(8)=B(1)*B(7)
1 B(9)=B(2)*B(7)
1 B(10)=B(3)/B(1)
1 B(11)=B(4)+B(14)
1 B(12)=B(1)*B(11)
1 B(13)=B(3)*B(11)
1 B(14)=B(12)-B(13)
1 B(15)=B(5)+B(14)
1 B(16)=SQRTF(B(15))
1 B(17)=B(6)*B(14)
1 M=H(5)
20 GO TO (30+40+501)=M
30 CONTINUE
40 WRITE OUTPUT TAPE 6*60*
1 B(1)*B(2)+B(3)+B(4)+B(24)
50 RETURN
60 FORMAT (1HE19.8,5E2u8/1M,1E19.8,5E2u8)
FREQUENCY 20(110+100)
END (0) 930TD000
HM=U*
DO 47 I=2*M1
DO 43 J=7*12
I
HA=ABS(H(I,J))
43
HM=MAX1F(HM,HA)
47
CONTINUE
30 D(9)=S(M3+2)+S(M3+4)
31 D(18)=HM
32 M5=M(5)
33
GO TO (60, 90, 130), M5
60 WRITE OUTPUT TAPE 6*70*((S(I,J)+J=1*4)*I=K+M3)
DO 65 J=1,10+3
I
J2=J+2
WRITE OUTPUT TAPE 6*80*
1((H(I,J)+M(I,J)+12)*J=J1+J2)*I=K+M11)
65
CONTINUE
70 FORMAT (1H0,E19.8,E20.8,E19.8,E320.8)
71 FORMAT (1H0,E19.8,E320,8/E19.8,E320.8)
80 FORMAT (1H0,E19.8,E520.8/E19.8,E520.8)
90 WRITE OUTPUT TAPE 6*120* M3*D(9)+D(18)
120 FORMAT (1H0,E19.8,F4.0)
130 IF(D(9))=170,170,140
140 DO 150 J=1,12
150 H(I,J)+M(I,J)
S(I+2)=S(30+2)
160 S(I+4)=S(30+4)
170 RETURN
FREQUENCY 100(0+1*7)+50(1+10+100)
END (0) 930TE00D
SUBROUTINE SRF
COMMON L,M,K,F1,F2,F3,F4,A,B,C
DIMENSION M(10),K(10),L(10),I(10),J(10),S(13),S(13)

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Approved for Public Release
MO=M(J)
R2=A(J)
220 DO 30 K=1,MO
230 R(I+J+K)=R2*R(I+J+K)
240 CONTINUE
S3=R(I+J+1)
S(I+3)=S3
C(I+1)=B(I+10)-S(I+2)/S1
C(I+9)=B(I+30)/S1
C(I+2)=S1*B(I+2)
C(I+10)=S3*B(I+32)
I C(I+3)=C(I+1)*C(I+2)
I C(I+4)=B(I+4)*C(I+2)
I C(I+5)=B(I+5)*C(I+3)
RETURN
FREQUENCY 110(20)*140(20)*
1 170(20)*180(10)*220(20)
END (0) 930TF000
SUBROUTINE SRG
C FORMAL AND RATIONAL EXPANSIONS
C M(4) MUST EXCEED 3
C R(1,2±1) MUST EQUAL 1
C POLE ON REAL AXIS BETWEEN TERMINAL POINTS NOT COVERED
C USES SRH THRU SRK
C
DIMENSION L(6),M(10),N(13),Q(16)
1 R(30,±29),L(30,4),G(29,29),I(12,29)
I DIMENSION H(31,12),P(24,6,30),Q(16,4,8)
1 B(20),C(30,8),D(9)
2 U(6),V(6)
3 A(-1,±1),D(1),D(1),P(1),P(1)
COMMON L,M,N,H,P,Q,R,S,C,D,E
I=L(1),L2=L(2)
1 C2=C1(1±2)
10 GO TO (20±10,10,2)
20 T1=R(1,1±2)
I PO=H(1±1)
I UD=H(1±4)
I VO=H(1±5)
I NO=H(1±6)
I PU=PO±UD
I UU=UD±UD
I AN=PO±8-B(15)
I DO=PU±8-B(15)
I AI=PO
I DI=PU
I A2=PO±PO
I DZ=PU±PU
I BS=8±8
I DD=31
I J=1±6
I P(1±1)±H(1±1)
I3 Q(1±1)±Q(1±1)
I L(4)±2
CALL SRH

930TG005
930TG010
930TG015
930TG020
930TG025
930TG030
930TG035
930TG040
930TG045
930TG050
930TG055
930TG060
930TG065
930TG070
930TG075
930TG080
930TG082
930TG085
930TG090
930TG092
930TG094
930TG095
930TG096
930TG097
930TG098
930TG100
930TG102
930TG105
930TG107
930TG110
930TG112
930TG115
930TG120
930TG125
930TG130
930TG135
930TG140
L(4)=K1  
L(5)=K1+1  
TN=TN+1  
CALL SRH  
CALL SRI  
A1(1)=A1(1)+T1  
A2(1)=A2(1)+T1  
A3(A1)=A3(A1)+1  
A3=A3#V(1)+A1#V(2)+V(3)  
A5=A2*(A0+A3)  
P1=AA/A5  
P2=-V(2)+A1#P1  
P3=-V(2)+A1#P2+AO#P1  
P(1+1*K1)=P1  
P(1+2*K1)=P2  
P(1+3*K1)=P3  
V(4)=V(4)-P1#UU  
V(5)=V(5)-P1#V0-U0#P2  
V(6)=V(6)-P1#V0-U0#P3  
D1(1)=D1(1)+T1  
D2(1)=D2(1)+T1  
D3=D3#2-B5  
D4=D3#V(4)+D1#V(5)+V(6)  
D5=D2+D0+D3  
P4=D4/D5  
P5=-V(4)+D2#P4  
P(1+4*K1)=P4  
P(1+5*K1)=P5  
P(1+6*K1)=-V(5)+D1#P5+D0#P4  
DO 60 J1=1*6  
J2=J1+6  
Q(J1*K1)={U(J1)+TN#P(1+1*K1)}  
60 Q(J1*K3)=U(J2)+TN#P(1+1*K2)  
P4=ABSF(P4)  
P4=MAXF(P4,P4)  
IF [PA(1)>(1*OE+15)]  
IF [PM(1)>UE+05]*PA(1)]  
CONTINUE
100  CO=1,0E-05
   GO TO 200
110  R0=1*0/R(I+1,11)
   DO 120 J=1,6
1120  P(I+J+1)=H(I+J)
   L(4)=1
   L(5)=2
   CALL SRJ
   I  Q(1+1)=V(J)+B(5)
   I  Q(2+J)=V(J)
   I  Q(3+J)=V(J)+C(1+5)
   DO 125 J=4,6
1125  Q(I+J)+=V(J)
   CALL SRH
   DO 130 J=1,6
   J2=J+6
   P(I+J+2)=R0#U(J1)
130  P(I+J+32)=R0#U(J2)
   L(4)=2
   L(5)=3
   CALL SRJ
   I  Q(1+2)=V(J)
   I  Q(2+2)=V(J)+C#P(I+1+1)
   I  Q(3+2)=V(J)+C(#1+4)
   I  Q(4+2)=V(J)
   I  Q(5+2)=V(J)+C#P(I+4+1)
   I  Q(6+2)=V(J)
   CALL SRH
   R1=0.5*R0
   DO 140 J=1,6
   J2=J+6
   P(I+J+3)=-R1#U(J1)
140  P(I+J+33)=-R1#U(J2)
   PA=ABS(P(I+4+3))
   PM=PA(1)
   MO=M(4)-1
150  DO 190 K=3,M0
   K1=K+1
   K2=K1+30
270 M5=M(5)
280 GO TO (290,310,320)*M5
290 WRITE OUTPUT TAPE 6*XWC
   ((P(1*J+K)*P(1*J+K+30)*J=1*3)*K=1*4)
300 WRITE OUTPUT TAPE 6*X300,
   ((P(1*J+K)*P(1*J+K+30)*J=4*6)*K=1*4)
310 CONTINUE
320 CALL SRK
      RETURN
      FREQUENCY 10(146),
  1   50(271)*70(100+0+1)*80(100+0+1)*
  2   150(126)*170(100+0+1)*80(100+0+1)*
  3   210(131)*230(29)*280(1+10+100)
END (0) 930TG800
SUBROUTINE SRH
C FIRST SUMMATION
C 1=1L(2)+2
C
DIMENSION L(6),M(10),N(30+4),
R(30+2*29)+S(30+4)+G(29+29)+O(2*29)
DIMENSION M(31+12)+P(30+6+30)+Q(6+48)+
B(20)+C(30+8)+D(19)+
U(6)+
I=1L(1)
N1=L(4)+L(2)-2
N2=M1+1
M3=N2+1
I=U(1)=(0+0+0)
10 IF (L(1)=M1+1) 20,20,30
20 L2=L(4)
R0=R(1+1L4)
U2=R0*C(1+2)
U1=R0*C(1+10)
GO TO 40
40 I=U2=(0+0+0)
30 DO 45 J=3+6
45 U1J=(0+0+0)
M2=M1+M1
50 DO 70 K2=3+30
70 CONTINUE
80 DO 100 K2=1+M2
20 K1=3+K1
R1=R(1+1K1)*FLOAT(K1-1)
DO 60 J1=1+K1
60 DO 80 J1=1+K1
80 CONTINUE
M2=M1+M1
100 CONTINUE
900TH010
900TH020
900TH030
900TH040
900TH050
900TH060
900TH070
900TH080
900TH090
900TH100
900TH110
900TH120
900TH130
900TH140
900TH150
900TH160
900TH170
900TH180
900TH190
900TH200
900TH210
900TH220
900TH230
900TH240
900TH250
900TH260
900TH270
900TH280
900TH290
900TH300
900TH310
900TH320
900TH330
900TH340
900TH350
900TH360
K2=K1+48
R2=R(1+2*K1)
DO 90 J1=1,6
J2=J1+6
U(2)*U(J1)-R2*Q(J1*K1)
90 U(J2)=U(J2)-R2*Q(J1*K2)
100 CONTINUE
RETURN
FREQUENCY 10:1,2,5;4:1,8;1:10
END (0) 930TH000
SUBROUTINE SRI
C
SECOND SUMMATION
C
L(I) MUST EXCEED 2
C
DIMENSION L(6),M(10),N(13U+4)
1
DIMENSION R(3U+2*29)+S(3U+4I)+Q(2*29)
I
DIMENSION M3(12),P(3U+6*30),Q(6+48)
1
B(20)=C(30+8)+D(19)+
2
U(6)+V(6)+
COMMON L+M+N+H+P+Q+R+S+G+O+B+C+D+U+V
I=L(1)
L3=L(3)
L5=L(5)
I
C2=C(1+2)
I
V(1)=U(1)
I
V(2)=U(2)+C2*P(I+1+L3)
I
V(3)=U(3)
I
V(4)=U(4)
I
V(5)=U(5)+C2*P(I+4+L3)
I
V(6)=U(6)
10
DO 30 K=3*2+L3
K2=L5-K
I
P1=P(I+1+K1)
I
P4=P(I+4+K1)
I
P7=P1+P4
I
DO 20 J=1+3
J2=J1+3
1
P0=P(I+J1*K2)
I
V(J1)=V(J1)-P1*P0
I
V(J2)=V(J2)-P4*P0-P7*P(I+J2*K2)
20
CONTINUE
RETURN
FREQUENCY 10(13)
END 930T1000
SUBROUTINE SRJ
C THIRD SUMMATION
C
DIMENSION L(6),M(10),N(30*4),
1 R(30*2*29),S(30*4),G(29*29),U(2*29)
I
DIMENSION H(31*12),P(30*6*30),Q(6*48),
1 B(201),C(30*8),D(99),
2 U(6),V(6)
COMMON L,M,N,H,P,Q,R,S,G0,B,C,D,U,V
I=L(1)
L4=L(4)
L5=L(5)
I
C3=C(1+3)
I
P2=P(1+2*L4)
I
P5=P(1+5*L4)
I
V(1)=P2
I
V(2)=P(1+3*L4)*C3*P(1+1*L4)
I
V(3)=P2*B(5)
I
V(4)=P5
I
V(5)=P(1+6*L4)*C3*P(1+4*L4)
I
V(6)=P5*B(5)
I
10 DO 30 K1=1,L4
K2=L5-K1
I
P1=P(1+K1)
I
P4=P(1+4*K1)
I
P7=P(1+7*K1)
I
10 DO 20 J1=1,3
J2=J1+3
20 CONTINUE
I
P0=P(1+J1*K2)
I
V(J1)=V(J1)+P1*P0
I
V(J2)=V(J1)+P4*P0+P7*P(1+J2*K2)
I
120 CONTINUE
I
RETURN
FREQUENCY 10(15)
END (0)
DO 60 K=1,N5
60 B(J)=A(J*K)+Y(K)
160 B(J)=B(D)
CALL SRL
DO 80 J=1,N5
80 Y(J)=Y(J)+X(J)
1 P1=M(W(1))
90 DO 110 J=2,N3
110 P0=M(J)
NO=M1N0F(J,N4)-1
DO 100 K=1,N0
KO=J-K
100 P0=P0+M(K)*W(J)
1 P1=P1+P0
1110 PI=M(J0,J1)+P0
1 P1+J0,N5)=1+0*(1
1 Q1=(1+0)
1 E0=M(J0)
DO 120 K=1,N5
120 K0=N2-K
1 K1=K+N6
1 Y0=Y(K)
1 Q1=Q1+Y0
1 EQ=E0+M(K0)*Y0
1260 P1=J0,J1)=Y0
1 F1=P1/Q1
1 Q0=ABS(F(1)+ABS(F1)
1 Z1+J0,P1
1 Z1+J0+Q1
1 Z1+J0+Q1+Q0
1 H1+J0)+F1
130 M1=J0+6=EO/J1+Q0
M5=M5
140 GO TO (130,160,170)*M5
150 DO 155 J1=1,4*3
155 J2=J1+2
WRITE OUTPUT TAPE 6+18U (((P1+J0,J1)+P1+J0,J1+J2)+K=1)*N3
WRITE OUTPUT TAPE 6+18U (((P1+J0,J1)+P1+J0,J1+J2)+K=N5+N2)
WRITE OUTPUT TAPE 6+18U (((Z1+J0,J1+J0,J1+J2)+J1+J2)+L=1)*X3)
355 CONTINUE
360 CONTINUE
380 FORMAT (3H1*E19.9*5E2.8/1H*E19.8*5E2.4,1H)
480 FREQUENCY 10(30),90(30),140(1,10,00)
END 930K0000

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SUBROUTINE SRL
C LINEAR SYSTEM SOLUTION
C SOLVES A*X = B
C 3**N*(1,4)**17 ONLY
C C DETA(I) MUST NOT VANISH
DIMENSION L(6),M(10),P(30),Q(31),R(16)
1 I = (16+16)+B(16)*X(16)
2 C(16+17)
COMMON L,M,N,P,A+B,X
LO=L(1)
KO=N(L0+4)
J0=KO-1
I0=J0-1
DO 20 J=J0+J0
20 K=J+J0
DO 110 J=J0+J0
110 C(J*K)=A(J*K)
120 C(J*K)=B(J)
DO 50 I=J+I0
50 I=I+1
1 C0M=(1.0+U)/C(J1)
DO 40 K=J1+K0
40 C1=C0*M(C1*K)
1 C1*K)=C1
DO 30 J=J1+J0
30 C(J*K)=C(J*K)-C(J1)*C1
40 CONTINUE
50 CONTINUE
DO 55 J=J1+J0
55 X(J)=C(J*K)
1 X(J0)=X(J0)+C(J0,J0)
DO 70 I=J+I0
70 K=K0-1
K1=K-1
I X1=X(K)
DO 60 J=J1+K1
60 X(J1)=X(J1)-C(J1)*X1
MAIN PROGRAM RPM
PROPER VALUE LOCATION
USES SRR THRU SRL
PLUS SRR THRU SRO
USES 10 DATA CARDS FOR BASIC FLOW
2***M11***29 NUMERATOR TERMS (BASIC FLOW)
2***M12***29 DENOMINATOR TERMS (BASIC FLOW)
PLUS 2 DATA CARDS FOR SOLUTION CONTROL
4***M41***30 MAXIMUM TERMS (PER EXPANSION)
3***M51***3 PRINTING (MIN ONLY)
PLUS 6 DATA CARDS FOR INTERPOLATION CONTROL
2***K11***16 POINTS (INITIAL INTERPOLATION)
2***K22***8 POINTS (FINAL INTERPOLATION)
1***K33***2 UNKNOWN (STREAM FREQUENCY)
1***K44***7 ROOTS (PER POINT)
1***K55***7 ITERATIONS (ROOT EXTRACTION)
1***K66***2 PRINTING (MAXIMI)
PLUS 2 DATA CARDS FOR INTERPOLATION RANGE
T(11) RADIUS (INITIAL INTERPOLATION)
930TM125
T(21) RADIUS (FINAL INTERPOLATION)
930TM125
PLUS 6 DATA CARDS FOR PARAMETERS (PER POINT)
A(11) STREAMWISE FREQUENCY (REAL PART)
930TM145
A(22) STREAMWISE FREQUENCY (IMAG PART)
930TM150
A(33) CROSSWISE FREQUENCY (REAL)
930TM155
A(44) TIMEWISE FREQUENCY (REAL)
930TM160
A(55) REYNOLDS NUMBER (REAL PART)
930TM165
A(66) REYNOLDS NUMBER (IMAG PART)
930TM170
2***K11***4 YIELDS JUST INITIAL INTERPOLATION
DIMENSION L(6) M(10), N(30), 41
1 R(31+2*29)+S(31+4)+G(29+29)+0(12+29)
930TM220
1 DIMENSION H(31,12),P(3*6*30),Q(6*48),
   2 B(20),C(30*8),D(9),
   3 U(6),V(6),
   4 T(2),A(6),
   5 E(6),
1 DIMENSION X(2,16),Y(3*16),Z(3,7),
   1 W(2,10),B(11),B2(11),B3(11),
   COMMON L*MNMPNSXSRSG0B0BC0DC0USV*,
   1 K*XY*Z
 CALL SRB
 M(5)=3
 READ INPUT TAPE 5*10* (K(1),I=1,6)
10 FORMAT (112)
   K1=K(1),
   K2=K(2),
   K3=K(3),
   K0=3*K3-2,
   IF (K1-5) 13,15,15
13 L0=1
   GO TO 17
15 L0=2
17 L1=K1-1
 READ INPUT TAPE 5*20* (T(I),I=1,2)
20 FORMAT (E12.8)
 WRITE OUTPUT TAPE 6*25* M(4) G(K(1),I=1,5)*(T(I),I=1,2)
25 FORMAT (1H0+119/H0*119*4120/1H0*E19.8*E20.8)
   DO 40 I=1,L0
   J0=K(1)
   TO=T(I)
   C0=1.0/FLOAT(J0)
   CJ=6.2831514*C0
   DO 30 J1=1,J0
   J2=J1+16
   CJ2=CI*FLOAT(J1-1)
   CJ3=COS(C1)
   CJ4=SIN(C2)
   W(I,J1)=TO*C3
930T225
930T230
930T235
930T240
930T245
930T247
930T250
930T255
930T260
930T265
930T270
930T275
930T280
930T285
930T290
930T295
930T300
930T302
930T304
930T305
930T306
930T307
930T308
930T309
930T310
930T315
930T317
930T318
930T320
930T325
930T330
930T335
930T340
930T345
930T350
930T355
930T360
930T365
930T370
WRITE OUTPUT TAPE 6 $I$,*9$ (2$1$+$J$)*Z(1$+$J$)*X(1$=1;3$)+$J$=1;4$)
WRITE OUTPUT TAPE 6 $I$,*9$ (1$+$E$)*I$,*1$=1;6$)
140 FORMAT (1$H$,*$E$1$9$;$5$E$2$U$8$;1$H$,*$E$1$9$;$5$E$2$U$8$)
GO TO 50
END (O) 930TM000
SUBROUTINE SRN
C POLYNOMIAL COEFFICIENTS
C UNIT CIRCLE DATA
C
DIMENSION L(6),M(10),N(30+6),R(30+20),S(30+6)+G(29+29),O(2+29)
1 1
DIMENSION H(31+12)+P(30+6+30)+Q(6+48)+
1 1
B(20)+C(30+8)+D(9),
2 U(6),V(6)
2
DIMENSION K(9)
2
DIMENSION X(2+16),Y(3+16)
COMMON L,M,N,R,P,Q,S,G,O,B,C,D,U,V,
1 X,X,Y
1
10=K(7)
JO=K(10)
DO 20 I=1,JC
20 RC=0+0)
DO 10 J=1,JO
K0=2MODF(1-1)+J-1+1
10 CO=CO+X(10*K0)*Y(1+J)
110 Y(2+1)=CO
RETURN
END (0) 930TN000
SUBROUTINE SRQ  

C  SIMPLE ZEROs NEAR ORIGIN  
C  
DIMENSION L(6),M(10),N(3U+4)  
1  R(30x2,29),S(30x4x4),T(30x2,29)xO(12x29)  
I  DIMENSION H(21x12),P(3u8x30),Q(6x68)  
1  B(20)xC(30x8)xD(9),  
2  U(81)xV(81)  
DIMENSION K(9)  
I  DIMENSION X(2x16),Y(3x16),Z(3,7)  
1  E(3x16)  
COMMON L,M,N+M,H+R+S+G+0+B+C+D+U+Y  
1  K+X+Y+Z  
K5*K(5)  
K6*K(6)  
K7*K(7)  
K8*K(8)  
K9*K(9)  
K10*K8+1  
K11*K7  
DO 10 J=1,K8  
110  E(1,J)=Y(3,J)  
DO 110 J=1,K9  
10  N0*K10=11  
N1=N0-1  
N2=N1-1  
20  IF (N0=3) 30 40 60  
130  Z(71,:)=E(1,1)/E(1,2)  
GO TO 120  
140  E(2,2)=E(1,3)  
1  E(3,1)=E(2,2)  
1  Z0=0*x+0  
DO 50 I=1,K5  
50  E(2,1)=E(1,2)+Z0*E(2,2)  
1  E2=E(1,1)+Z0*E(2,1)  
1  E3=E(2,1)+Z0*E(3,1)  
150  Z0=Z0-E2/E3  
GO TO 90  
930T0010  
930T0020  
930T0030  
930T0040  
930T0050  
930T0060  
930T0070  
930T0080  
930T0090  
930T0100  
930T0110  
930T0120  
930T0130  
930T0140  
930T0150  
930T0160  
930T0170  
930T0180  
930T0190  
930T0195  
930T0200  
930T0210  
930T0220  
930T0230  
930T0240  
930T0250  
930T0260  
930T0270  
930T0280  
930T0290  
930T0300  
930T0310  
930T0315  
930T0320  
930T0330  
930T0340  
930T0350  
930T0360  
930T0370
160 E(2x1)=E(1xNO)
1  E(1xN2)=E(2xN1)
  Z0=(0+0+1)
  DO 80 I=1xK5
  1  E(2xN2)=E(1xN1)+Z0*E(2xN1)
     DO 70 J=2xN2
        0=NO-J
        J1=J0+1
     1  E12+J0)=E(1xJ1)+Z0*E(2xJ1)
170  E13+J0)=E(2xJ1)+Z0*E(3xJ1)
  1  E2=E(1,1)+Z0*E(2,1)
180  E3=E(2,1)+Z0*E(3,1)
  10 Z0=Z0-E2/E3
  190 Z1(K7,i1)=Z0
  DO 100 J=1xN1
1100 E(1xJ)=E(2+J)
110 CONTINUE
120 GO TO (130+160)*K6
130 WRITE OUTPUT TAPE 6x14,47 ((Y(J+1),Y(i+J+16),J=1+3,J+1+K11)
140 FORMAT (1H0*E19x*8*E2U*8/I1H*E19x*8*5E2U*8))
  150 WRITE OUTPUT TAPE 6x15,47 ((Z(K7,J)+Z(K7,J+7),J=1xK9)
  160 RETURN
FREQUENCY 20(1+1+100)+120(1+100)
END (0) 930T0000
930T0380
930T0390
930T0395
930T0400
930T0410
930T0420
930T0430
930T0440
930T0450
930T0460
930T0470
930T0480
930T0490
930T0500
930T0510
930T0520
930T0530
930T0540
930T0550
930T0560
930T0570
930T0580
930T0590
930T0600
930T0610
PART 2

STABILITY OF COMPRESSIBLE FLOW

OVER A FLAT PLATE

W. Byron Brown
<table>
<thead>
<tr>
<th>Dimensionless Quantities</th>
<th>Characteristic Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ gas density</td>
<td>$\frac{\rho}{\rho_0}$</td>
</tr>
<tr>
<td>$T$ gas temperature</td>
<td>$\frac{T}{T_0}$</td>
</tr>
<tr>
<td>$\mu_1$ first viscosity coefficient</td>
<td>$\frac{\mu_1}{\mu_{10}}$</td>
</tr>
<tr>
<td>$\mu_2$ second viscosity coefficient</td>
<td>$\frac{\mu_2}{\mu_{20}}$</td>
</tr>
<tr>
<td>$\sigma$ disturbance wave number</td>
<td>$\sigma^{-1}$</td>
</tr>
<tr>
<td>$c$ phase velocity of the disturbance</td>
<td>$\frac{c}{c_0}$</td>
</tr>
<tr>
<td>$\gamma$ specific heat ratio</td>
<td>$\frac{c_p}{c_v}$</td>
</tr>
<tr>
<td>$R$ Reynolds number</td>
<td>$\frac{\rho_0 U_0}{\mu_{10}}$</td>
</tr>
<tr>
<td>$M$ Mach number</td>
<td>$\left(\frac{R* \text{gas constant per gram}}{\sqrt{\gamma \rho_0 R^*}}\right)$</td>
</tr>
<tr>
<td>$\mathcal{M}$ Mach number</td>
<td>$\mathcal{M} = M \cos \gamma$</td>
</tr>
<tr>
<td>$\sigma$ Prandtl number</td>
<td>$\frac{c_p \mu_{10}}{E_1}$</td>
</tr>
<tr>
<td>$\psi$ angle between main velocity and the disturbance velocity</td>
<td>$\frac{c_p \mu_{10}}{E_1}$</td>
</tr>
<tr>
<td>$k$ thermal conductivity</td>
<td>$\frac{c_p \mu_{10}}{E_1}$</td>
</tr>
<tr>
<td>$L$ length unit</td>
<td>$\frac{x^*}{R}$</td>
</tr>
<tr>
<td>$x^*$ distance from the stagnation point</td>
<td>$\frac{x^*}{R}$</td>
</tr>
<tr>
<td>Dimensionless Quantities</td>
<td>Characteristic Measure</td>
</tr>
<tr>
<td>-------------------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>(x)</td>
<td>non-dimensional distance ( \ell )</td>
</tr>
<tr>
<td>(y)</td>
<td>distance from the flat plate ( \delta )</td>
</tr>
<tr>
<td>(w)</td>
<td>undisturbed velocity parallel to the plate ( U_0 )</td>
</tr>
<tr>
<td>(v)</td>
<td>undisturbed velocity component in boundary layer \perp \text{wall} ( U_0 )</td>
</tr>
<tr>
<td>(f)</td>
<td>velocity disturbance amplitude in (x) direction</td>
</tr>
<tr>
<td>(q)</td>
<td>velocity disturbance amplitude in (y) direction</td>
</tr>
<tr>
<td>(\eta)</td>
<td>amplitude of the pressure disturbance</td>
</tr>
<tr>
<td>(\tau)</td>
<td>amplitude of the density disturbance</td>
</tr>
<tr>
<td>(\theta)</td>
<td>amplitude of the temperature disturbance</td>
</tr>
</tbody>
</table>

A bar over a quantity denotes average value. A prime denotes differentiation with respect to \(y\). Subscript \(0\) denotes free-stream value; subscripts \(r\) and \(i\) denote real and imaginary parts.
The exact numerical solution of the Lee-Lin equations (Reference 1) at Mach 5.8 showed better agreement with experimental data (Reference 2) than previous approximate solutions (References 3 and 4), but it was still 25% low at the critical Reynolds number and differed much more with the upper branch data. An improved calculation therefore was attempted by dropping the usual assumption that the flow was parallel to the flat plate and that the velocity component of the mean flow perpendicular to the plate could be safely neglected.

These calculations restored to the system of stability equations the terms involving the velocity component of the mean flow perpendicular to the flat plate. Calculations made with the more complete equations (References 5 and 6) showed that the expected increase in critical Reynolds number was much too large and that, in order to produce an agreement with experimental data, the three-dimensional aspect of the disturbance velocities would have to be taken into account. This was done by the method suggested by Dunn (Reference 3). This report gives the new equations and the new results. Dunn considers the Lee-Lin equations as obtained from the complete three-dimensional set (3 momentum equations instead of 2) by a transformation in direction so that the flow makes an angle $\gamma$ with the $x$-axis. Then

$$\tilde{N} = M \cos \gamma$$
$$\tilde{\sigma} \tilde{N} = \sigma M$$
$$\tilde{\sigma} \tilde{R} = \sigma R$$
$$\tilde{c} = c$$  \hspace{1cm} (1)

The solution of the equations two dimensional in form yields eigenvalues for $\tilde{\sigma}$ and $\tilde{R}$, once a value of $\tilde{N}$ has been chosen. The values of $\sigma$ and $R$ (corresponding to the real flow) are found from the transformation equations (1).
II. ANALYSIS

The disturbance forms assumed are those of Dunn, Reference 6, namely

\[ u_1 = w(y) + f(y) \exp \left[ i (\alpha_1 x + \alpha_3 z - \alpha_1 t) \right] \]
\[ u_2 = v(y) + \alpha_1 \phi(y) \exp \left[ i (\alpha_1 x + \alpha_3 z - \alpha_1 t) \right] \]
\[ u_3 = h(y) \exp \left[ i (\alpha_1 x + \alpha_1 t) \right] \]
\[ s = p(y) + s(y) \exp \left[ i (\alpha_1 x + \alpha_3 z - \alpha_1 t) \right] \]
\[ s = p(y) + \pi(y) \exp \left[ i (\alpha_1 x + \alpha_3 z - \alpha_1 t) \right] \]
\[ T = \Xi(y) + \Theta(y) \exp \left[ i (\alpha_1 x + \alpha_3 z - \alpha_1 t) \right] \]

\[ \mu_1(y) + \frac{\partial \Theta}{\partial t}(y) \exp \left[ i (\alpha_1 x - \alpha_3 z - \alpha_1 t) \right] \]

In Reference 6, these are substituted into the equations of motion and reduced to the two-dimensional form given by the transformation

\[ \alpha = \alpha_1 f + \alpha_3 h \]
\[ \phi = \gamma \]
\[ \beta = \alpha_1 + \alpha_3 \]

A. EQUATIONS

By Reference 7 (page 37) \( \mu_2 = -\frac{2}{T} \mu_1 \). This substitution has been made in all the equations. Then the first momentum equation (x-direction) becomes

\[ e \left[ \left( \frac{u - c}{T} \right) + \frac{8}{5} \frac{\mu}{R} \alpha^2 \right] + e' \left( \psi - \frac{T'}{R} \frac{du}{dt} \right) \]
\[ + \psi' \left( \frac{\alpha}{T} - i \frac{T'}{R} \frac{d\alpha}{dt} \right) + \frac{\pi}{T} \left( \nu \frac{u^2}{2} + \frac{1}{2} g \right) \]
\[ + e \left( \frac{w'}{T^2} - i \frac{10}{9} \frac{\nu \phi}{R} \frac{du}{dt} + \frac{\psi}{R} \frac{du}{dt} - \frac{\psi}{R} \frac{T'}{dt} \frac{d^2u}{dt^2} \right) \]
\[ + \theta \left( - \frac{w'}{R} \frac{du}{dt} \right) = \frac{T}{R} \frac{\mu}{R} + \psi' \left( - \frac{u^2}{9R} \right) \]

\[ (2) \]

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The second momentum equation (y-direction) is

\[\begin{align*}
\frac{\partial}{\partial t} \left( \frac{10}{9} \frac{T'^2}{R} \frac{\partial \mu}{\partial T} \right) & + \frac{\partial}{\partial T} \left( - \frac{10}{R} + \frac{10}{9} \frac{\mu}{R} \right) \\
+ \frac{\partial}{\partial T} \left( \frac{10}{T} \frac{\partial \mu}{\partial T} \right) & + \frac{\partial}{\partial T} \left( \frac{10}{9} \frac{\mu}{R} \right) \\
= & \frac{\partial}{\partial T} \left( \frac{8}{9} \frac{T'^2}{R} \frac{\partial \mu}{\partial T} \right) + \frac{\partial}{\partial T} \left( \frac{8}{9} \frac{\partial \mu}{\partial T} \right) - \frac{n'}{\mu'}
\end{align*}\]

\[\text{(3)}\]

The energy equation is

\[\begin{align*}
f \left( \frac{10}{9} \frac{T'^2}{R} \right) & + \frac{\partial}{\partial T} \left( \frac{10}{R} \frac{\partial \mu}{\partial T} \right) \\
+ \frac{\partial}{\partial T} \left( \frac{10}{9} \frac{\mu}{R} \right) & + \frac{\partial}{\partial T} \left( \frac{10}{T} \frac{\partial \mu}{\partial T} \right) \\
= & \frac{\partial}{\partial T} \left( \frac{8}{9} \frac{T'^2}{R} \frac{\partial \mu}{\partial T} \right) + \frac{\partial}{\partial T} \left( \frac{8}{9} \frac{\partial \mu}{\partial T} \right) \\
- & \frac{\partial}{\partial T} \left( \frac{10}{9} \frac{\partial \mu}{\partial T} \right)
\end{align*}\]

\[\text{(4)}\]

The continuity equation is

\[\begin{align*}
f' \left( \frac{10}{9} \frac{T'}{R} \right) & + \frac{\partial}{\partial T} \left( \frac{10}{R} \frac{\partial \mu}{\partial T} \right) \\
+ \frac{\partial}{\partial T} \left( \frac{10}{9} \frac{\mu}{R} \right) & + \frac{\partial}{\partial T} \left( \frac{10}{T} \frac{\partial \mu}{\partial T} \right)
\end{align*}\]

\[\text{(5)}\]

The equation of state has been used to replace \( r \) and \( r' \) in the continuity equation

\[\begin{align*}
r = \frac{10}{T} \frac{\partial}{\partial T} \\
r' = \frac{10}{T} \left( \frac{\partial}{\partial T} \right)
\end{align*}\]

\[\text{(6)}\]

When \( n' \) in the second momentum equation is replaced by \( n' \) as obtained by differentiating the continuity equation, the term \( \frac{n'}{\mu'} \) occurs. This term is dropped as negligibly small (\( \mu \) changes but very little through the boundary layer). Thus a system of 6 first order differential equations may be obtained.

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Approved for Public Release
B. SOLUTION OF THE EQUATIONS

These equations are solved in the same manner as the abbreviated equations of Reference 8, except that the characteristic equation is more complicated so that it has to be solved numerically rather than by formulae.

In order to write these in the standard form, six linear first order equations, the following substitutions are made

\[ Z_1 = \xi \]
\[ Z_2 = \xi' = Z'_1 \]
\[ Z_3 = \varphi \]
\[ Z_4 = -\frac{\eta}{H^2} \]
\[ Z_5 = \theta \]
\[ Z_6 = \theta' = Z'_6 \]

Boundary conditions are
\[ Z_1 = Z_2 = Z_3 = 0 \]
when
\[ y = 0 \] and \[ Z_1, Z_3, Z_5 \] bounded as \[ y \to \infty \]

These may be written

\[ \sum a_{ij} Z_j = \sum b_{ij} Z_j \quad (i = 1, 2, \ldots, 6) \]

where the row index is assigned to the six equations as follows

<table>
<thead>
<tr>
<th>i</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ Z'_1 = Z_2 ]</td>
</tr>
<tr>
<td>2</td>
<td>First momentum (2)</td>
</tr>
<tr>
<td>3</td>
<td>Continuity (5)</td>
</tr>
<tr>
<td>4</td>
<td>Second momentum (3)</td>
</tr>
<tr>
<td>5</td>
<td>[ Z'_5 = Z_6 ]</td>
</tr>
<tr>
<td>6</td>
<td>Energy (4)</td>
</tr>
</tbody>
</table>
The \( a_{ij} \) and \( b_{ij} \) are found from the preceding equations. They are

\[
\begin{align*}
a_{11} &= 1 \\
a_{22} &= \frac{M}{R} \\
a_{23} &= \frac{1}{\nu} y^2 \\
a_{33} &= -1 \\
a_{34} &= -\frac{\nu y^2}{\omega} \\
a_{41} &= -\frac{1}{9} \frac{8}{\omega} - \frac{M}{R} \\
a_{43} &= -\frac{8}{\omega} \frac{\omega}{T} + \frac{8}{\omega} \frac{\omega}{T} \frac{d\omega}{dT} + \frac{8}{\omega} \frac{\omega}{T} \left[-(\omega T)\right] \\
a_{44} &= \frac{8}{9} \frac{\omega}{T} \frac{\omega}{T} \left[-(\omega T)\right] + i_\omega (w-c) + v' \left[1 + \frac{\nu y^2}{\omega} \right] - \frac{1}{\gamma} \\
a_{45} &= \frac{8}{9} \frac{\omega}{T} \frac{\omega}{T} \left[-(\omega T)\right] - i_\omega (w-c) - v' \\
a_{46} &= \frac{8}{9} \frac{\omega}{T} \frac{\omega}{T} \\
a_{55} &= 1 \\
a_{63} &= -\frac{\gamma (\gamma-1)}{\omega} + \frac{10}{9} \frac{\omega}{\omega} (\gamma (\gamma-1)) R^2 \\
\end{align*}
\]

\[
\begin{align*}
a_{66} &= \frac{M}{R} \\
b_{12} &= 1 \\
b_{21} &= \frac{8}{9} \frac{\omega}{R} \frac{\omega}{R} \\
b_{22} &= \frac{8}{9} \frac{\omega}{R} \frac{\omega}{R} \\
b_{23} &= \frac{8}{9} \frac{\omega}{R} \frac{\omega}{R} \\
b_{24} &= \frac{8}{9} \frac{\omega}{R} \frac{\omega}{R} \\
b_{25} &= \frac{8}{9} \frac{\omega}{R} \frac{\omega}{R} \\
b_{26} &= \frac{8}{9} \frac{\omega}{R} \frac{\omega}{R} \\
b_{31} &= 1 \\
b_{33} &= -(\omega T)\left[-(\omega T)\right] \\
b_{34} &= \frac{M}{R} \left[-(\omega T)\right] + i_\omega (w-c) + v' \\
b_{35} &= \frac{1}{\omega} \left[-(\omega T)\right] - i_\omega (w-c) - v' \\
\end{align*}
\]

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\[ b_{36} = \frac{-v}{2T} \]
\[ b_{41} = \frac{10}{9} \frac{\Gamma R d \theta}{R d T} \]
\[ b_{42} = \frac{10}{9} \]
\[ b_{43} = \frac{10}{9} \left( w-c \right) + \frac{\omega \gamma}{2} + \frac{w^3}{2} \frac{\phi}{R} - \frac{\Delta}{2} \frac{\alpha}{R} \left[ \frac{-(\Delta T)^2}{R} \right] \]
\[ b_{44} = v \frac{\nu}{v^2} \frac{\phi}{R} + \frac{\Delta}{2} \left[ \frac{v}{R} \left( \alpha \left( \Delta T \right) \right)' - v \left( \Delta T \right)' + 10 \omega' + v' \right] \]
\[ b_{45} = -v \frac{\nu}{v^2} \frac{\phi}{R} \frac{d \theta}{d T} - \frac{\Delta}{2} \frac{v}{R} \frac{d \theta}{d T} + \frac{\alpha}{R} \left[ \frac{T'}{2} \right] \left[ 2v \left( \Delta T \right)' - 10 \omega' - v' \right] \]
\[ + \frac{1}{2} \left[ 2v \left( \Delta T \right)' + 2v \left( \Delta T \right)' - 10 \omega' - v' \right] \]
\[ b_{46} = - \frac{\Delta}{2} \frac{v}{R} \frac{d \theta}{d T} + \frac{\alpha}{R} \left[ \frac{T'}{2} \right] \left[ v' \right] + v \left( \Delta T \right)' \]
\[ b_{56} = 1 \]
\[ b_{61} = \frac{1}{\nu} \left( \gamma-1 \right) + \frac{20}{9} \frac{\nu}{\gamma} \left( \gamma-1 \right) \frac{\nu}{R} \left( \gamma-1 \right) \frac{\nu^2}{R} \]
\[ b_{62} = -2w' \frac{\nu}{R} \left( \gamma-1 \right) \frac{\nu^2}{R} \]
\[ b_{63} = \frac{\nu}{R} \left( \Delta T \right)' - 10 \omega' \frac{\nu}{R} \left( \gamma-1 \right) \frac{\nu^2}{R} \]
\[ b_{64} = v \frac{\nu}{v^2} \left( \Delta T \right)' + \left( \gamma-1 \right) \frac{\nu}{R} \left( \gamma-1 \right) \frac{\nu^2}{R} \]
\[ b_{65} = \frac{1}{\nu} \left( w-c \right) - \frac{\nu}{R} \left( \Delta T \right)' - \frac{\Delta}{2} \frac{\nu}{R} \left( \gamma-1 \right) \frac{\nu^2}{R} \left( \gamma \frac{v}{v^2} + w^2 \right) \frac{d \theta}{d T} \]
\[ + \frac{\nu}{R} \frac{\nu}{R} \frac{v}{v^2} \frac{d \theta}{d T} + \frac{\Delta}{2} \frac{v}{R} \frac{d \theta}{d T} \frac{w^2}{R} \frac{d \theta}{d T} \]
\[ b_{66} = \frac{1}{\nu} \left( \gamma \frac{v}{v^2} + w^2 \right) \frac{d \theta}{d T} \]

In order to obtain the \( Z_j \), the system

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{22} & a_{23} & 0 & 0 & 0 \\
0 & 0 & a_{33} & a_{34} & 0 & 0 \\
0 & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & a_{63} & 0 & 0 & a_{66}
\end{bmatrix}
= 
\begin{bmatrix}
z'_{1} \\
z'_{2} \\
z'_{3} \\
z'_{4} \\
z'_{5} \\
z'_{6}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\
b_{31} & 0 & b_{33} & b_{34} & b_{35} & b_{36} \\
b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\
0 & 0 & 0 & 0 & 0 & 1 \\
b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66}
\end{bmatrix}
\begin{bmatrix}
z_{1} \\
z_{2} \\
z_{3} \\
z_{4} \\
z_{5} \\
z_{6}
\end{bmatrix}

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must be transformed into a system

\[ Z_i = \sum_{j=1}^{n} c_{ij} Z_j \quad (i = 1, 2, \ldots, n) \]

which may be achieved by simple elimination procedures. We have

\[ Z_i' = Z_2 \]
\[ Z_i' = \frac{1}{a_{22}} (b_{2j} Z_j - a_{23} Z_j') \]
\[ Z_j' = \frac{1}{a_{33}} (b_{3j} Z_j - a_{34} Z_4) \]
\[ Z_4' = \frac{1}{a_{44}} (b_{4j} Z_j - a_{43} Z_3 - a_{45} Z_5 - a_{46} Z_6) \]
\[ Z_5' = Z_5 \]
\[ Z_6' = \frac{1}{a_{66}} (b_{6j} Z_j - a_{63} Z_3) \]

For solution outside the boundary layer, the determinant of the characteristic equation is now

\[
\begin{vmatrix}
-\lambda & 1 & 0 & 0 & 0 & 0 \\
C_{21} & C_{22} - \lambda & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} - \lambda & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} - \lambda & C_{45} & C_{46} \\
0 & 0 & 0 & 0 & -\lambda & 1 \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} - \lambda \\
\end{vmatrix} = 0
\]

Expansion of this determinant yields the characteristic equation:

\[
\lambda^6 - \lambda^5 \left( C_{33} - C_{44} + C_{66} + C_{22} \right) + \lambda^4 \left( -C_{21} - C_{22} \left( C_{33} + C_{44} + C_{66} \right) + C_{33} C_{44} + C_{33} C_{66} + C_{44} C_{66} \right) - C_{66} - C_{66} C_{34} - C_{66} C_{44} - C_{66} C_{66} - C_{46} C_{66} - C_{23} C_{32} - C_{24} C_{32} - C_{26} C_{26} \\
\lambda^3 \left( C_{21} C_{33} + C_{22} C_{44} + C_{22} C_{66} \right) - C_{22} C_{33} C_{44} - C_{22} C_{33} C_{66} - C_{22} C_{44} C_{66} - C_{22} C_{44} C_{44} - C_{22} C_{44} C_{66} - C_{22} C_{66} C_{66} \\
\lambda^2 \left( -C_{21} C_{44} - C_{21} C_{66} - C_{22} C_{44} - C_{22} C_{66} - C_{22} C_{44} C_{66} - C_{22} C_{66} C_{66} \right) \]

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The six complex roots \( \lambda_s \) (s = 1, 2, \ldots, 6) are found by a numerical method.

Then outside the boundary layer where the coefficients are constant, the solution of the system is (Reference 14)

\[
Z_1 = \sum_{s=1}^{6} \bar{k}_{1s} \left( \lambda_s e^{kEy} \right)
\]

where the \( \bar{k}_{1s} \) are the cofactors of the elements of the fourth row of the characteristic determinant outside the boundary layer \( y = b \) to \( y = w \).

These values are

\[
\bar{k}_{1s} = \begin{vmatrix}
C_{65} & \lambda_s & C_{66} - \lambda_s \\
C_{24} & \lambda_s & C_{34} - C_{33} + C_{23} C_{34} \end{vmatrix}
\begin{vmatrix}
(C_{33} + \lambda_s) \bar{k}_{1s} \\
C_{24} C_{63} - C_{23} C_{64} + (C_{25} + C_{26}) \lambda_s \end{vmatrix}
\]

\[
\bar{k} - \lambda_s \bar{k}_{1s}
\]

\[
\bar{k}_{3s} = \begin{vmatrix}
\lambda_s & C_{32} + C_{31} & C_{34} & C_{35} + \lambda_s & C_{36} \\
\lambda_s & C_{42} + C_{41} & C_{44} - \lambda_s & C_{45} + \lambda_s & C_{46} \\
\lambda_s & C_{62} + C_{61} & C_{64} - \lambda_s & C_{65} + \lambda_s & C_{66} - \lambda_s^2
\end{vmatrix}
\]

\[
\bar{k}_{4s} = \begin{vmatrix}
\lambda_s & C_{32} + C_{31} & C_{33} - \lambda_s & \lambda_s & C_{36} \\
C_{44} & \lambda_s & C_{42} + C_{41} & C_{43} & C_{46} \\
C_{63} & \lambda_s & C_{62} + C_{61} & C_{64} & C_{65} + \lambda_s & C_{66} - \lambda_s^2
\end{vmatrix}
\]

\[
\bar{k}_{5s} = \begin{vmatrix}
C_{33} - \lambda_s & C_{34} & \lambda_s & C_{32} + C_{31} \\
C_{43} & C_{44} - \lambda_s & \lambda_s & C_{42} + C_{41} \\
C_{63} & C_{64} & \lambda_s & C_{62} + C_{61}
\end{vmatrix}
\]

The rest of the solution is exactly like that of Reference 1.

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III. CALCULATIONS

In the solution of equations 2 to 6, the velocity profiles parallel
to the plate were taken from Reference 9 except for Mach 8, which was com-
puted by the method of Reference 10. The perpendicular profiles were com-
pared by the equation of Reference 9.

\[ \rho v^2 = \frac{1}{2} \frac{1}{R} \left( \rho v^2 \int_0^\eta \frac{\eta}{0} d\eta \right) \]  

(7)

In our non-dimensional notation, this becomes

\[ \nu R = \frac{1}{2} \left( \nu \int_0^\eta \frac{\eta}{0} d\eta \right) \]  

(8)

The viscosity variation was computed by the use of Sutherland’s equation
as given in Reference 9.

\[ 10^5 \mu^* = \frac{1.658 \ T^*^{ \frac{3}{2}}}{T_0 + 116.4} \]  

(9)

The non-dimensional form is therefore

\[ \frac{110.4}{1 + \frac{T_0}{T_m} \frac{110.4}{1 + \frac{T_0}{T_m}}} \]  

(10)

The constant in the formula, \( \frac{T_0}{T_m} \), depends on the freestream temperature
and must be altered when this changes. Here \( T_m \) must be in degrees Kelvin.
The Prandtl number was assumed constant through the layer. If the pressure
change through the boundary layer is neglected, then

\[ \rho T = 1 \]  

(11)

Stability calculations were made for \( M = 2.2 \) and \( \bar{M} = 2.2, 1.474, 1.232, 1.6852 \), corresponding to angles with the main flow of \( 0^0, 48^0, 50^0 \) and \( 60^0 \) respectively.

For \( M = 3 \), \( \bar{M} \) was 3, 4.33, 3.3, 2.5, 3.0, 1.294, corresponding to angles of \( 0^0, 30^0, 45.5^0, 60^0, 53^0 \) and \( 75^0 \) respectively.

For \( M = 8 \), \( \bar{M} \) was 8, 5.6, 4, corresponding to angles of \( 0^0, 45.5^0 \) and \( 60^0 \).

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IV. RESULTS AND DISCUSSION

A comparison at Mach 5 of neutral stability curves computed by the Dunn-Lin equations, the Lees-Lin equations and the Dunn-Cheng equations is shown in Figure 1. The critical Reynolds number based on momentum thickness turned out to be 185 for Dunn-Lin, 325 for Lees-Lin. When the terms involving the velocity component perpendicular to the flat plate were included, the critical \( R_e \) jumped to 1250. The experimental value of Reference 2 was about 550. When these last equations were used at a wave angle of 60°, \( R_e \) critical became 643. None of the computed curves agreed very well with the data points on the upper branch, though the 60° curve was much closer than the others.

The directional effect is shown in Figure 2, where the critical Reynolds number is plotted against the wave angle.

When similar calculations were made for Mach 2.2, the results are shown in Figure 3. The Lees-Lin equations gave lower branch values of \( R_e \) about 65 below the data points of Reference 12 and upper branch values about 18% below the data points. When the Dunn and Cheng equations were used with a wave angle of 50°, the computed lower branch came almost exactly on the data points, while the upper branch fell about 10% below the data points.

A few results at Mach 8 are shown in Figure 4. The abscissa here is \( R_e = \sqrt{\frac{U^2}{V} \cdot \frac{R_e}{P_{10}}} \). The Lees-Lin curve shows values of \( R_e \) around 600, the Dunn and Cheng around 5040, and the Dunn and Cheng with a wave angle of 45.5° about 1900. Also shown are some experimental transition measurements from Reference 13. These latter are somewhat (14%) above the directional computations.

V. CONCLUSIONS

The inclusion in the stability equations of the boundary layer velocity component and the allowance for a three-dimensional disturbance velocity improve the agreement between the observed and calculated neutral stability curves.


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FIGURE 1  OBSERVED AND COMPUTED NEUTRAL STABILITY CURVES.
Figure 2: Critical Reynolds No. vs. Direction, N = 5
Figure 3: Observed and Computed Neutral Stability Curves
Mach No. 2.2
PART 3

NUMERICAL SOLUTION OF THE COMPLETE THREE DIMENSIONAL

STABILITY EQUATIONS OF THE COMPRESSIBLE

BOUNDARY LAYER ON A FLAT PLATE

W. Byron Brown
<table>
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<th>Characteristic Measure</th>
</tr>
</thead>
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<td>gas density</td>
</tr>
<tr>
<td>$T$</td>
<td>gas temperature</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>first viscosity coefficient</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>second viscosity coefficient</td>
</tr>
<tr>
<td>$c$</td>
<td>phase velocity of the disturbance</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>specific heat ratio</td>
</tr>
<tr>
<td>$R$</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>disturbance wave number</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>disturbance wave number</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Prandtl number</td>
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<tr>
<td>$\theta$</td>
<td>angle between main velocity and the disturbance velocity</td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity</td>
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<td>$L$</td>
<td>length unit</td>
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<tr>
<td>Dimensionless Quantities</td>
<td>Characteristic Measure</td>
</tr>
<tr>
<td>-------------------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>x*</td>
<td>distance from the stagnation point</td>
</tr>
<tr>
<td>x</td>
<td>non-dimensional distance</td>
</tr>
<tr>
<td>y</td>
<td>distance from the flat plate</td>
</tr>
<tr>
<td>w</td>
<td>undisturbed velocity parallel to the plate layer</td>
</tr>
<tr>
<td>v</td>
<td>undisturbed velocity component in boundary layer perpendicular to the wall</td>
</tr>
<tr>
<td>f</td>
<td>velocity disturbance amplitude in x direction</td>
</tr>
<tr>
<td>φ</td>
<td>velocity disturbance amplitude in y direction</td>
</tr>
<tr>
<td>h</td>
<td>velocity disturbance amplitude in z direction</td>
</tr>
<tr>
<td>π</td>
<td>amplitude of the pressure disturbance</td>
</tr>
<tr>
<td>r</td>
<td>amplitude of the density disturbance</td>
</tr>
<tr>
<td>θ</td>
<td>amplitude of the temperature disturbance</td>
</tr>
</tbody>
</table>

A bar over a quantity denotes average value. A prime denotes differentiation with respect to y. Subscript o denotes free-stream value; subscripts r and i denote real and imaginary parts.
I. INTRODUCTION

It has been shown (Reference 1) that the approximate method (Reference 2) of acoustics for the three-dimensional aspect of the disturbance velocities is of only limited value and does not agree well with experimental data on the upper branches of the neutral stability curves, especially at Mach 3.

To remedy this defect and to obtain a calculation method that is more reliable at high Mach numbers, a direct solution of the linearized stability equations that contain all three momentum equations for the disturbance velocity has been obtained. The new calculations and the new results for Mach 2.2 and Mach 5 are given in this report.

II. ANALYSIS

The disturbance forms assumed are those of Dunn, Reference 2, namely

\[ u_1 = u(y) + \varepsilon(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

\[ u_2 = v(y) + \sigma_1^0(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

\[ u_3 = \eta(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

\[ p = p(y) + \tau(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

\[ p = p(y) + \sigma_2(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

\[ T = T(y) + \vartheta(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

\[ \mu_1 = \omega_1(y) + \frac{\partial}{\partial T} \vartheta(y) \exp \left( i (a_{1x} + a_{2y} - c_1 t) \right) \]

A. EQUATIONS

These are the same as in Reference 1, except that a third momentum equation is included; also another disturbance wave number, as in Reference 3. Thus the equations to be solved are now:

First momentum (x direction):

\[ f \left( \frac{21}{4} \frac{\partial^2}{\partial T^2} + \omega_1^2 - c - i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \varepsilon^2 \left( \frac{\partial^2}{\partial T^2} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \varepsilon^1 \left( \frac{\partial^2}{\partial T^2} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \varepsilon^0 \left( \frac{\partial^2}{\partial T^2} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) \]

\[ + \vartheta \left( \frac{\partial^2}{\partial T^2} + i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \vartheta^1 \left( \frac{\partial^2}{\partial T^2} + i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \vartheta^0 \left( \frac{\partial^2}{\partial T^2} + i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) \]

Second momentum (y direction):

\[ f \left( \frac{10}{9} \frac{\partial^2}{\partial T^2} + \varepsilon^2 \left( \frac{\partial^2}{\partial T^2} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \varepsilon^1 \left( \frac{\partial^2}{\partial T^2} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \varepsilon^0 \left( \frac{\partial^2}{\partial T^2} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) \]

\[ + \vartheta \left( \frac{\partial^2}{\partial T^2} + i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \vartheta^1 \left( \frac{\partial^2}{\partial T^2} + i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) + \vartheta^0 \left( \frac{\partial^2}{\partial T^2} + i \frac{\partial}{\partial T} \frac{\partial}{\partial T} \frac{\partial}{\partial T} \right) \]
\[ + \varphi \left( \frac{\alpha_1^2}{T} (\omega - c) + \frac{\alpha_1 \nu' \varepsilon_0}{T} + \frac{\mu_0 \alpha_1}{M^2} \left( \frac{\nu' \varepsilon_0}{T} \right) \right) \]

\[ - \theta \left( - \frac{\nu' \varepsilon_0}{T^2} - \frac{\alpha_1 \nu' \varepsilon_0}{G} \frac{d \nu'}{dT} - \frac{8}{9} \frac{\nu' \varepsilon_0}{T} \frac{d \nu'}{dT} \right) + \varphi' \left( - \frac{8}{9} \frac{\nu' \varepsilon_0}{G} \frac{d \nu'}{dT} \right) \]

Third momentum (z direction):

\[ \frac{1}{T} \left( \omega - c \right) \delta \varphi = \frac{1}{2} \frac{\nu' \varepsilon_0}{G} \frac{d \nu'}{dT} + \frac{h}{T} \frac{d \nu'}{dT} \left( T - \frac{R_0}{T} \right) - \frac{h}{R} \left( \alpha_1^2 + \alpha_2^2 \right) \]

Energy equation is:

\[ \frac{1}{T} \left( \alpha_1 (\gamma - 1) + \frac{2}{9} \frac{\nu' \varepsilon_0}{G} (\gamma - 1) \right) \frac{d \nu'}{dT} \]

\[ + \varphi' \left( \alpha_1 (\alpha T) + \frac{2}{9} \frac{\nu' \varepsilon_0}{G} (\gamma - 1) \right) \frac{d \nu'}{dT} - \frac{\mu_0 \alpha_1}{M} \left( \gamma (\gamma - 1) \frac{d \nu'}{dT} \right) \]

\[ + \theta \left( \frac{\alpha_1}{T} (\omega - c) - \frac{\nu' \varepsilon_0}{T} \left( \frac{8}{9} \nu' + \frac{1}{2} \nu' \right) \frac{d \nu'}{dT} \right) \]

Continuity equation is:

\[ f_i - \varphi' \left( \alpha_1 (\gamma - 1) + \frac{16}{9} \frac{\nu' \varepsilon_0}{G} (\gamma - 1) \right) \]

\[ + \frac{\partial}{\partial T} \left( \gamma (T) + \frac{8}{9} \alpha_1 (\omega - c) + \frac{2}{9} \frac{\nu' \varepsilon_0}{G} \right) \]

\[ - \theta \left( \frac{\alpha_1}{T} (\omega - c) - \nu' \right) \frac{d \nu'}{dT} = - \varphi' \frac{\alpha_1}{\varphi} \]

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The equation of state has been used to replace \( r \) and \( r' \) in the continuity equation:

\[
\begin{align*}
    r &= \frac{\pi}{\tau} - \frac{\beta}{\tau^2} \\
    r' &= \frac{\nu^2 \left( \frac{\pi}{\nu^2} \tau - \frac{\pi}{\nu^2} \tau' \right)}{\nu^2} - \frac{\nu^2 \tau'^2 - 2\beta \tau'}{\nu^2}
\end{align*}
\]  

(6)

When \( \phi' \) in the second momentum equation is replaced by \( \phi'' \) as obtained by differentiating the continuity equation, the term \( \tau' \) occurs. Since both \( \tau' \) and \( \phi' \) are very small, this term is negligible and is dropped. Thus a system of 8 first order differential equations may be obtained.

B. SOLUTION OF EQUATIONS

This is done by substituting

\[
\begin{align*}
    z_1 &= \xi \\
    z_2 &= \xi' = z_1' \\
    z_3 &= \psi \\
    z_4 &= \frac{\pi}{\nu^2} \\
    z_5 &= \tau \\
    z_6 &= \tau' = z_5' \\
    z_7 &= h \\
    z_8 &= h' = z_7'
\end{align*}
\]

(7)

This system is set up as follows:

\[
[a][z'] = [b][z]
\]

<table>
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<tr>
<th>Row Index</th>
<th>Equation</th>
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<tr>
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<td>First momentum</td>
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<td>3</td>
<td>Continuity</td>
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<td>4</td>
<td>Second momentum (modified by ( \phi'' ) replacement)</td>
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<td>5</td>
<td>( z_4' = z_6 )</td>
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<td>( z_7' = z_8 )</td>
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<tr>
<td>8</td>
<td>Third momentum</td>
</tr>
</tbody>
</table>

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The $a_{ij}$ and $b_{ij}$ matrices are then as follows:

\[ a_{11} = 1 \]

\[ a_{22} = -\frac{c}{k} \]

\[ e_{23} = \frac{-lw}{9R} \]

\[ a_{33} = -1 \]

\[ a_{34} = -\frac{\nu \mu}{\sigma_1} \]

\[ a_{41} = -\frac{8}{9} \frac{\mu}{R} \]

\[ a_{43} = \frac{8}{9} \frac{\mu}{R} (\ln T)' + \frac{\sigma_1 k}{\nu} + \frac{8}{9} \frac{\mu}{R} \frac{d\mu}{dT} \]

\[ a_{44} = -\frac{1}{Y} \left( -\frac{8}{9} \frac{\mu}{R} \right) \left( \nu (\ln T)' + \sigma_1 (\nu - c) + 2\nu' \right) \]

\[ a_{45} = \frac{8}{9} \frac{\mu}{RT} \left( \nu (\ln T)' - \sigma_1 (\nu - c) - \nu' \right) \]

\[ a_{46} = \frac{8}{9} \frac{\mu}{\Delta R} \]

\[ a_{47} = -\frac{8}{9} \frac{\nu \mu}{R} \]

\[ a_{55} = 1 \]

\[ a_{63} = -c_1 (\gamma - 1) + \frac{16}{9} \frac{\nu \mu \nu (\gamma - 1)}{R} \]

\[ a_{66} = \frac{\sigma_1 k}{\nu} \]

\[ a_{77} = 1 \]

\[ a_{88} = \frac{\mu}{R} \]

\[ b_{12} = 1 \]

\[ b_{21} = \frac{1}{T} \left( \nu - c \right) + \left( \frac{2}{\sigma_1} + \frac{1}{\nu} \right) \frac{\mu}{R} \]
\[ b_{22} = \frac{\mu}{T} \frac{T'}{R} \frac{d\mu}{dT} \]
\[ b_{23} = \frac{\sigma_1^2}{T} \frac{1}{R} \frac{d\mu}{dT} \]
\[ b_{24} = \frac{\alpha_1}{\nu} \frac{\nu \nu'' \nu'}{T} \]
\[ b_{25} = -\frac{\nu''}{R} \frac{d^2 \nu}{dT^2} - \frac{\nu''}{R} \frac{d\nu}{dT} - \frac{\nu' \nu''}{T} + \frac{10}{9} \frac{\nu' \nu'' \nu}{R} \frac{d\nu}{dT} \]
\[ b_{26} = -\frac{\nu''}{R} \frac{d\nu}{dT} \]
\[ b_{31} = i \]
\[ b_{33} = -\left(\nu \nu'' \nu' \right) \]
\[ b_{34} = \frac{\mu}{\alpha_1} \left[ -\nu \left(\nu \nu'' \nu' \right) + \nu' \right] \]
\[ b_{35} = \frac{1}{\alpha_1} \left[ 2\nu \left(\nu \nu'' \nu' \right) + \nu_1 \left(\nu - \nu'' \right) - \nu' \right] \]
\[ b_{36} = \frac{\nu'}{\alpha_1} \]
\[ b_{37} = \frac{i \nu''}{\alpha_1} \]
\[ b_{41} = \frac{10}{9} \frac{T' \nu_1}{R} \frac{d\mu}{dT} \]
\[ b_{42} = \frac{i \mu}{\sigma_1^2} \]
\[ b_{43} = \frac{i \sigma_1^3}{T} \left(\nu - \sigma_1^2 \right) - \frac{8}{9} \frac{\sigma_1^2}{R} \left(\nu \nu'' \nu' \right) + \frac{\sigma_1^3}{R} \frac{d\mu}{dT} \]
\[ b_{45} = \frac{\mu}{9} \frac{d\mu}{dT} \left[ \frac{1}{T} \left[ 2\nu \left(\nu \nu'' \nu' \right) + \alpha_1 \left(\nu - \nu'' \right) - \nu' \right] - \frac{\nu' \nu'' \nu'}{T} \right] \]
\[ + \frac{1}{T} \left[ 2\nu \left(\nu \nu'' \nu' \right) + 2\nu \left(\nu \nu'' \nu' \right) - \nu_1 \left(\nu - \nu'' \right) - \nu' \right] \]
\[ - \frac{i \nu''}{R} \frac{d\nu}{dT} - \frac{6 \nu''}{9} \frac{d\nu}{dT} \]
\[ -118 - \]
\[ b_{60} = \frac{8}{9} \mu_0 \left( \frac{\nabla T}{T} - \nu \right) - \frac{2}{9} \frac{\nabla T}{R} \frac{d\mu}{dt} \]

\[ b_{66} = 1 \]

\[ b_{61} = 1 \sigma_1 (\gamma - 1) + 1 \frac{20}{9} \frac{\mu}{R} \gamma (\gamma - 1) \frac{H^2}{R} \]

\[ b_{62} = - \frac{2w' \mu_0 (\gamma - 1) \mu^2}{R} \]

\[ b_{63} = \sigma_1 (\ln \tau)' - \frac{21w' \mu_0^2 (\gamma - 1) H^2}{R} \]

\[ b_{64} = \nu H^2 (\ln \tau)' + (\gamma - 1) H^2 \nu \]

\[ b_{65} = \frac{1 \sigma_1 (w-c)}{T} \frac{\nabla T}{R} \frac{d\mu}{dt} - \frac{\nabla T^2}{\sigma R} \frac{d^2 \mu}{dt^2} + \frac{\nu \mu}{\sigma R} \left( \sigma_1^2 + \sigma_3^2 \right) - \frac{\nu}{R} (\ln \tau)' \]

\[ b_{66} = \frac{\nu}{R} \frac{21w' \mu_0^2 (\gamma - 1) H^2}{\nabla T} \left( \nu^2 + \frac{\nabla T}{9} \right) \frac{d\mu}{dt} \]

\[ b_{67} = \frac{1}{R} (\gamma - 1) \sigma_3 \]

\[ b_{76} = 1 \]

\[ b_{64} = \frac{1 \sigma_3}{\gamma} \]

\[ b_{87} = \frac{1 \sigma_1 (w-c)}{T} + \left( \sigma_1^2 + \sigma_3^2 \right) \frac{\nu}{R} \]

\[ b_{88} = - \frac{T}{R} \frac{d\mu}{dt} + \frac{\nu}{T} \]

The system

\[ [\mu] [x'] = [b] [x] \]

is reduced to a system

\[ [1] [x'] = [c] [x] \]

by multiplying the right hand side by the reciprocal \([\mu]^{-1}\). Thus \([c] = [\mu]^{-1} [b] \).
Once the $[C]$ matrix is computed, the characteristic equation is found and solved numerically. The rest of the solution is essentially the same as that used in Reference 4. The solution is carried out in two parts. If the boundary layer depth is $b$ (point where the boundary layer velocity is 99.5% of the free stream value), then within this distance the $C_{ij}$ coefficients are variables. Hence the integration from $y = 0$ to $y = b$ is carried out numerically. When $y > b$, the $C_{ij}$'s are constant. Hence the solution is the sum of eight exponential terms, one for each root of the characteristic equation. Since $Z_1$, $Z_2$, $Z_3$ and $Z_4$ are bounded as $y \to \infty$, the coefficients of four of these terms, those in which the real part of the root is positive, must vanish. At the point $y = b$, the numerical solutions must of course match the exponential solutions. Thus four conditions must be satisfied here for the four coefficients to vanish. These suffice to determine the eigen values required. Specifically in this case, the numerical solutions are

$$Z_1 = \sum_{j=1}^{8} C_j Z_1^{(j)} \quad (i = 1, 2, \ldots, 8)$$

The $Z_1^{(j)}$ are fundamental solutions defined by their initial conditions

$$Z_1^{(j)}(0) = b_{ij}$$

$$b_{ij} = 0 \text{ if } i \neq j, \quad b_{ij} = 1 \text{ if } i = j$$

The initial conditions ($y = 0$) are

$$Z_1 = 0, \quad Z_2 = 0, \quad Z_3 = 0, \quad Z_4 = 0, \quad Z_5 = 0, \quad Z_6 = C_6, \quad Z_7 = 0, \quad Z_8 = 0$$

Thus

$$Z_1(0) = C_1 Z_1^{(1)}(0) = C_1 = 0$$

$$Z_3(0) = C_3 Z_3^{(3)}(0) = C_3 = 0$$

$$Z_5(0) = C_5 Z_5^{(5)}(0) = C_5 = 0$$

$$Z_7(0) = C_7 Z_7^{(7)}(0) = C_7 = 0$$

Hence the general solution is

$$Z_1 = Z_1^{(2)} + C_4 Z_1^{(4)} + C_6 Z_1^{(6)} + C_8 Z_1^{(8)}$$
and the problem is to determine $C_0$, $C_6$, $C_8$ and any two of the real parameters $a$, $b$, $R$, $c_\nu$, $c_i$ from the conditions at $y = b$, $K_5 = K_6 = K_7 = K_8 = 0$.

The general exponential solution can be written

$$z_i = \sum_{s=1}^{8} \bar{K}_{is} (K_s e^{\lambda y}) \quad (i = 1, 2, \ldots, 8) \quad (12)$$

where the $\bar{K}_{is}$ are the cofactors of the elements of the fourth row of the characteristic determinant

$$\det (C_{ij}^* - \lambda = \bar{K}_{ij}) = C_{ij}^* \text{ (values of } C_{ij}^* \text{ when } y > b)$$

Each of these is computed numerically by the machine.

Since Equations (12) applied at $y = b$ form a system of simultaneous linear equations for evaluating the 8 $K_i$'s, upon solving and setting $K_8 = 0$ $s = 5, 6, 7, 8$ four homogeneous linear functionals in the $Z_i$'s result which must be satisfied when $y = b$.

Thus the boundary conditions can be written

$$\sum_{i=1}^{8} \bar{K}_{ij} Z_j (b) = 0 \quad i = 5, 6, 7, 8 \quad (13)$$

where the matrix $X_{ij}$ is the inverse of the matrix $\bar{K}_{ij}$.

Thus, at $y = b$

$$\sum_{j=1}^{8} \bar{K}_{ij} Z_j^{(2)} + C_6 \sum_{j=1}^{8} \bar{K}_{ij} Z_j^{(4)} + C_6 \sum_{j=1}^{8} \bar{K}_{ij} Z_j^{(6)} + C_8 \sum_{j=1}^{8} \bar{K}_{ij} Z_j^{(8)} = 0 \quad (14)$$

Let $X_{ij}^*$ denote the linear functional

$$\sum_{j=1}^{8} X_{ij} Z_j (b)$$

and

$$f_{ie}^{(p)} = \sum_{j=1}^{8} K_{ij} Z_j^{(p)} (b)$$

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Then Equation (14) can be written

$$k_i^* = k_i^{(2)} + C_A k_i^{(4)} + C_B k_i^{(6)} + C_C k_i^{(8)} = 0$$

where $i = 5, 6, 7, 8$

The first three of these can be used to compute $C_A$, $C_B$, and $C_C$. Substitution in the fourth yields a complex number for $k_i^*$. To make this number vanish, the parameters $C_1$, $C_2$, $R$, $c_f$, and $C_f$ must be adjusted. The result is a set of eigen-values. For a neutral curve, $C_1 = 0$, $C_2/C_f$ is given a value (usually to make $R$ a minimum) and $R$ and $c_f$ are computed for a series of values of $C_f$. The plot of $C_f$ against $R$ is the usual neutral curve.

III. CALCULATIONS

The calculations were carried out in the same manner as in Reference 1 except that, of course, equations 1 to 6 were used. These contain another parameter $C_3$ so the complete parameter list consists of $C_1$ (wave number), $R$ (Reynolds number), $c_f$ (wave velocity), $C_f$ (amplification or damping factor), $C_3$ (wave number). After testing a few cases where $C_1$ was fixed and $C_3$ varied, as in Figure 1, it was decided to adopt a fixed value of the ratio $C_3 = 1.428$.

This corresponds to a flow angle of 55°; i.e., $\tan 55° = 1.428$. This seems to be near the angle between the three-dimensional flow and the main flow where the critical Reynolds number is a minimum. (Reference 2 gives about 51° in a similar case for a lower Mach number.)

At this ratio, $C_3 = 1.428$, neutral stability curves were computed for Mach numbers 2.2 and 5.

IV. RESULTS AND DISCUSSION

The results for Mach number 2.2 are shown in Figure 2, where the data of Reference 5 are plotted also. Agreement between theory and observed data is quite good on both upper and lower branches of the neutral curve.

The Mach 3 results are shown in Figure 3, where the data of Reference 4 are plotted. Here also, agreement is good on both branches of the neutral curve.

V. CONCLUSIONS

The addition of the third momentum equation to the usual set of stability equations for supersonic laminar boundary layers gives good agreement with observed data for both the upper and lower branches of the neutral stability curve.
REFERENCES


FIGURE 1  EFFECT ON THE REYNOLDS NO. OF VARYING THE WAVE NO. RATIO

\[ a = 0.06 \quad M = 5 \]
Figure 3: Theoretical and Experimental Neutral Stability Curves

$M = 5$
Theoretical Investigations of Boundary Layer Stability

Final Report (July 1965 to August 1966)

Brown, W. Byron
Raies, Gibbs S.

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Air Force Flight Dynamics Laboratory
Research and Development Division
Wright-Patterson AFB, Ohio 45433

None

The mathematical analysis underlying a Fortran program for calculating the proper solutions of the Orr-Sommerfeld system with sufficient accuracy and economy for applying the resonance theory of transition is described. This program covers subsonic growths, rather than time scale growths as in previous computations, of mainly two-dimensional Fourier components of the motion. It employs various innovations providing as much accuracy from efficient single-precision arithmetic as could be obtained from awkward multi-precision arithmetic in previous calculation schemes. The source programs and some sample calculations, for the principal modes of oscillation of the Blasius boundary layer, are included.

The less-than stability equations for compressible flow have been extended to include the terms involving the component of the mean boundary layer flow perpendicular to the flat plate. At Mach 5 this more than doubled the critical Reynolds number. Allowance was then made for the three-dimensional aspect of the disturbance velocity. The final result was to give good agreement with observed data in the lower branch of the neutral stability curve at Mach 2.6 and Mach 5, fair agreement with the upper branch at Mach 2.6 and large discrepancies with the data in the upper branch at Mach 5.

Comparison of experimentally determined neutral stability curves with those computed by simplified approximations have disagreed considerably at high Mach numbers on the upper branch, even when agreement was fairly good on the lower branch. To improve the calculations, the complete set of three-dimensional (See Continuation Sheet)

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ABSTRACT

stability equations, including all three momentum equations and also the component of the mean flow in the boundary layer normal to the surface, are solved numerically. This set of equations can be reduced to a set of eight linear equations with complex coefficients. The theoretical solutions for Mach 2.0 and Mach 5 are compared with experimental data and show good agreement in both upper and lower branches.