THREE-DIMENSIONAL FINITE ELEMENT MODEL FOR LAKE CIRCULATION

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A solution technique is presented for the three-dimensional (3-D) circulation of water in shallow lake basins. The basis for solution is a finite element analogue to the shallow lake equations due to Eckman. Galerkin's method provides the minimization functionals for the derivation of rectangular parallelopiped element equations. Since the matrix equations include a nonlinear boundary condition linking the free-surface elevation to the interior elements, the Newton-Raphson Method is suggested as a solution method for the governing matrix equations.

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INTRODUCTION

To date, no complete understanding of the complicated phenomenon involved in the circulation of large lakes has been obtained. In order to gain some insight into this problem, it is necessary to simplify the dynamic equations of motion. It is then possible to obtain a crude prediçtion of the response of the lake to various factors, thereby developing in a systematic way a better understanding of a more complicated phenomenon of interest. However, analytical solutions to even these simpler equations are rare and often crude or unduly restrictive. It is the purpose of this paper to present a formulation of a three-dimensional finite element model for the circulation in shallow lakes, e.g. Lake Erie.

Simplified dynamical equations of motion describing massive water circulations were obtained by Ekman (1)* in 1905. Analytical solutions for idealized rectangular basins have been derived by Lai and Rumer (2) based on this theory. Cheng (3,4) has presented a finite element analogue for irregular basins. However, in those works depth averaging techniques were used and the problem was formulated in terms of the average stream function. This eliminated the essential nonlinearity involved in the free surface elevation and reduced the problem to that of a two-dimensional basin. Such a formulation has no meaning near the boundaries of the basin where spurious flow across the boundaries occur. In addition, any numerical solution based on derived variables such as the stream function implies less accurate results for the real variables, i.e., velocities, because of the necessity to differentiate numerically to obtain them. For these reasons a solution technique for the three

dimensional problem was sought.

Matrix equations based on the simplified three-dimensional Navier-Stokes, continuity, and boundary equations due to Ekman (1) are obtained by the use of a combination of the methods of finite elements (5) and of weighted residuals (6). The specific method of weighted residuals chosen was Galerkins' method (7) in which the approximating function itself is taken is the weighting function.

In the succeeding sections, the method of weighted residuals as used in a finite element context, will be developed and applied to the governing differential equations for massive lake circulation. A finite element analogue to these equations for an irregular lake basin using rectangular parallelopiped elements will then be presented. Since the resulting algebraic equations are nonlinear in nature, an iterative convergence method, the Newton-Raphson method, is used to develop a solution technique for these equations. An outline of the solution process is given along with a selection technique for the initial guess required for the Newton-Raphson method.

GALERKIN, FINITE ELEMENT METHOD

The method of weighted residuals (7) consists of assuming a trial solution to the governing equations in the form of a finite series of known functions of position multiplied by undetermined parameters, and requiring that these parameters be such that the field equations and boundary conditions be satisfied in some approximate sense. The manner in which the governing equations are satisfied determines the particular subclass of the method of weighted residuals used.

The subclass chosen for the work described herein is Galerkin's method which consists of using the approximating functions as the weighting functions. Consider the set of differential equations

$$D_i(u_j) = 0 \quad \bar{x}\, V \quad i,j = 1,2 \ldots M \tag{1}$$

with associated boundary conditions

$$u_j(\bar{x}) = \bar{u}_j \quad \bar{x}\epsilon S_u \tag{2a}$$

$$L(u_j) = 0 \quad \bar{x}\epsilon S_L \tag{2b}$$

where $L(\cdot)$ and $D_i(\cdot)$ denote differential operators involving spatial derivatives of the dependent variables u_j, and where V is the total domain of interest bounded by $S = S_u + S_L$.

If a trial solution of the form

$$u_j^* = \sum_{k=1}^{N} c_{kj}\, \phi_k\, (\bar{x}) \tag{3}$$

is chosen, where $\phi(\bar{x})$ are specified functions of $\bar{x}$ which satisfy the boundary conditions on S_u, then the residuals, or errors, in the differntial equations and boundary conditions on S_L can be constructed as follows:

$$R_i(c_{kj}, \phi_k, \bar{x}\epsilon V) = D_i(u_j^*) \tag{4a}$$

$$R_L(c_{kj}, \phi_k, \bar{x}\epsilon S_L) = L(u_j^*) \tag{4b}$$

If u_j^* had been the true solution, the residuals would be identically zero. Therefore, the best choice of the c_{kj}'s for a given set of functions ϕ_k is that which makes R_o and R_i minimum. Galerkin's method consists of making the residuals orthogonal to each of the approximating functions ϕ_k,

$$\int_V \phi_k R_i \, dV = 0 \qquad (5a)$$

$$\int_{S_L} \phi_k R_L \, dS = 0 \qquad (5b)$$

which leads to a set of algebraic equations for c_{kj}.

The rationale behind this criterion is as follows (7): the ϕ_k's are members of a complete set (convergence) and R_L and R_i are piecewise continuous functions; a fundamental property of a complete set is that each member of the set (ϕ_k) is orthogonal to a continuous function (R_L and R_i) only if that function is identically zero; therefore, if a finite number of members of a complete set are used as approximating functions and each member is made orthogonal to the residuals, then in the limit $N \to \infty$ the residuals are identically zero.

Galerkin's method, per se, cannot be easily applied to problems with complicated domains in that it is not feasible to derive approximating functions applicable over the total region and which also satisfy all boundary conditions of the form $u(\bar{x}) = \bar{u}$, $\bar{x} \epsilon S_u$. However, the use of Galerkin's method within the context of a finite element scheme allows one to use simple approximating functions within small regions and to replace boundary conditions with simple interface continuity conditions for interior elements by expressing the approximating solutions in terms of nodal values of the basic unknowns rather than in terms of arbitrary parameters c_{kj}.

It is possible if simple element formulations are used that not all of the interface continuity conditions are satisfied. Because of the approximate nature of the finite element method, and because of the possibility of successive refinements in the numerical modeling of the

region, it has been shown in many instances that acceptable results are still obtained despite the lack of satisfaction of all of the interface continuity conditions. Of course, it is possible to derive more complicated element formulations which satisfy a greater number of these continuity conditions through definition of additional element nodes or of additional nodal variables, e.g., nodal derivatives of the variables.

DERIVATION OF LAKE EQUATIONS

The governing differential equations for flow in a large lake are the Navier-Stokes equations in the three Cartesian directions x_i (a comma denotes differentiation with respect to the succeeding subscripts)

$$\rho \frac{Du_i}{Dt} - \sigma_{ij,j} - \rho F_i = 0 \qquad i = 1,2,3 \tag{6a}$$

the continuity equation

$$u_{j,j} = 0 \tag{6b}$$

and the constitutive relation between stresses σ_{ij} and the velocity gradients

$$\sigma_{ij} = - p\, \delta_{ij} + \epsilon\rho\, (u_{i,j} + u_{j,i}) \tag{6c}$$

where $\frac{Df}{Dt}$ = convective derivative = $\frac{\partial f}{\partial t} + u_j f_{,j}$

ρ = mass density, F_i, u_i = body force per unit mass and velocity components respectively in the x_i direction, p = hydrostatic pressure, and δ_{ij} = Kronecker delta and ϵ = vertical turbulent momentum transport coefficient (eddy viscosity). The use of the eddy viscosity as a crude measure of turbulence effects is justified only by the lack of a more accurate measure.

The boundary conditions for these field equations are

$$u_i = 0 \quad \text{on} \quad S_s \tag{7a}$$

$$u_j n_j = 0 \quad \text{on} \quad S_f \tag{7b}$$

$$\sigma_{ij} n_j = \tau_i \quad \text{on} \quad S_f \tag{7c}$$

$$\rho u_j n_j = q \quad \text{on} \quad S_q \tag{7d}$$

where S_s = solid boundary, S_f = free surface, S_q = inlet/outlet region, n_j = components of outward unit normal to the total boundary surface, $S = S_s + S_f + S_q$, q = flow rate per unit area of S_q, and τ_i = applied tractions on S_f.

The Rossby number, the ratio of inertial to Coriolis forces is small for currents observed in the Great Lakes. For this reason, the convective terms in Eq. 6a can be neglected. (8) Also, neglecting the horizontal components of the Coriolis force due to vertical motion, and assuming that the Coriolis parameter $f = 2\,\omega \sin \Phi$ is constant (ω = angular speed of rotation of the earth and Φ = mean latitude of the lake), one can write the Navier-Stokes equation as

$$\frac{1}{\rho}\,\sigma_{ij,j} + u_j e_{ij} f = 0 \qquad i = 1,2 \tag{8a}$$

$$\frac{1}{\rho}\,p_{,3} + \epsilon\, u_{3,jj} + \epsilon_{,3}\,(u_{3,j} + u_{j,3}) + g = 0 \tag{8b}$$

where e_{ij} = permutation tensor, and g = acceleration of gravity. If it is now assumed (1) vertical momentum transfer is negligible compared to the gravitational force, Eq. (8b) reduces to that of the assumption of hydrostatic pressure distribution

$$p = \rho\, g\, (\eta - x_3) \tag{8c}$$

where $\eta = \eta(x_1, x_2)$ is the free surface elevation above the datum $x_3 = 0$ on the mean surface of the lake.

In summary, the governing equations for massive circulation in large lakes are

$$\frac{1}{\rho} \sigma_{ij,j} + f\, e_{ij}\, u_j = 0 \qquad i = 1,2 \tag{9a}$$

$$u_{j,j} = 0 \tag{9b}$$

$$\frac{1}{\rho} \sigma_{ij} = -\, g(\eta - x_3) + \epsilon\, (u_{i,j} + u_{j,i}) \tag{9c}$$

along with the boundary conditions, Eqs. (7), of which Eq. 7b is of special interest. In terms of the free surface elevation η, Eq. 7b can be written as

which introduces a nonlinearity into the system of field equations.

An approximation to the solution of Eqs. (9) is taken in the form

$$u^*_i = \sum_k \phi_k C_{ki} \tag{10a}$$

$$\eta^* = \sum_k \phi_{ko} C_{ko} \tag{10b}$$

where ϕ_k, ϕ_{ko} are assumed functions of position, and C_{ki}, C_{ko} are unknown parameters. Equations (9) become

$$\frac{1}{\rho} \sigma^*_{ij,j} + f\, e_{ij}\, u^*_j = R_{1i} \qquad i = 1,2 \tag{11a}$$

$$u^*_{j,j} = R_{23} \tag{11b}$$

$$\frac{1}{\rho} \sigma^*_{ij} = -\, g(\eta^* - x_3)\delta_{ij} + \epsilon(u^*_{i,j} + u^*_{j,i}) \tag{11c}$$

$$u_1^* \eta^*_{,1} + u_2^* \eta^*_{,2} - u_3^* = R_{oo} \text{ on } x_3 = 0 \tag{11d}$$

where R_{ij} denotes the errors, or residuals, introduced by the approximations. Galerkin's criteria for this system of equations are

$$\int_V \phi_k, R_{ij}\, dV = 0 \qquad i = 1,2;\ j = 1,3 \tag{12a}$$

$$\int_{S_f} \phi_{ko} R_{oo}\, dS = 0 \tag{12b}$$

In order to recover the boundary conditions, Eqs. (7), from Eqs. (12), it is necessary to integrate Eqs. (12) by parts and to apply the divergence theorem (6). This yields

$$\int_V (\phi_{k,j} \sigma^*_{ij}/\rho - \phi_k f e_{ij} u^*_j)\, dv = \oint_S \phi_k \sigma^*_{ij} \eta_j\, dS \tag{13a}$$

$$\int_V \phi_{k,j} u^*_{,j}\, dv = \oint_S \phi_k u^*_j \eta_j\, dS \tag{13b}$$

$$\int_{S_f} \phi_{ko} (u^*_1 \eta^*_{,1} + u^*_2 \eta^*_{,2} - u^*_3) = 0 \tag{13c}$$

If Eqs. (7) and (11c) are substituted into the right hand sides of Eqs. (13), there results

$$\int_V [\epsilon\, \phi_{k,j}(u^*_{i,j} + u^*_{j,i}) - g\, \phi_{k,i}\, \eta^* - f\phi_k e_{ij} u^*_j]\, dV$$

$$= -\int_V g\, x_3\, \phi_{k,i}\, dV + \int_{S_f} \phi_k \tau_i/\rho\, dS + \int_{S_s+S_q} \phi_k \sigma^*_{ij} \eta_j\, dS \tag{14a}$$

$$\int_V \phi_{k,j} u^*_j\, dV = \int_{S_q} \phi_k\, q/\rho\, dS \tag{14b}$$

$$\int_{S_f} \phi_{ko} (u^*_1 \eta^*_{,1} + u^*_2 \eta^*_{,2} - u^*_3)\, dS = 0 \tag{14c}$$

where u_i^* and η^* are given by Eqs. (10) in terms of the unknown parameters C_{ki} and C_{ko}. Equations (14) constitutes a set of algebraic equations for those parameters and for the tractions $\sigma_{ij}^*\eta_j$ on the side and bottom boundaries of the lake basin.

It should be noted that the tractions $\sigma_{ij}^*\eta_j$ on $S_s + S_q$ cannot be determined from Eq. (11c) since that equation provides an inadequate description for the complex state of stress in the boundary region. Instead, as will be seen subsequently, the boundary tractions are determined directly from the finite element analogue to Eq. (14a) once the interior distribution of velocities has been determined. An analogy can be drawn to structural mechanics problems in which unknown tractions are specified at points of prescribed displacements.

FINITE ELEMENT ANALOGUE

Consider the lake to be subdivided into a finite number of rectangular parallelpiped elements interconnected at prescribed nodal locations (see Fig. 1). A two-dimensional rectangular grid of elements is superimposed at the datum $x_3 = 0$. This grid is to account for the free surface integral, Eq. 14c, and for the surface elevation η present in Eqs. (14a,b). In other words, each node of each volume element in the lake is considered to be connected through the η term with the corresponding node in the surface grid directly above the interior node.

The finite element analog to Eqs. (10) and (13) is obtained by identifying the unknown parameters C_{ki} and C_{ko} with nodal values of u_i^* and η^*, respectively. Thus

$$u_i^* = \frac{1}{b_3} [\psi][A] \quad U_i \tag{15a}$$

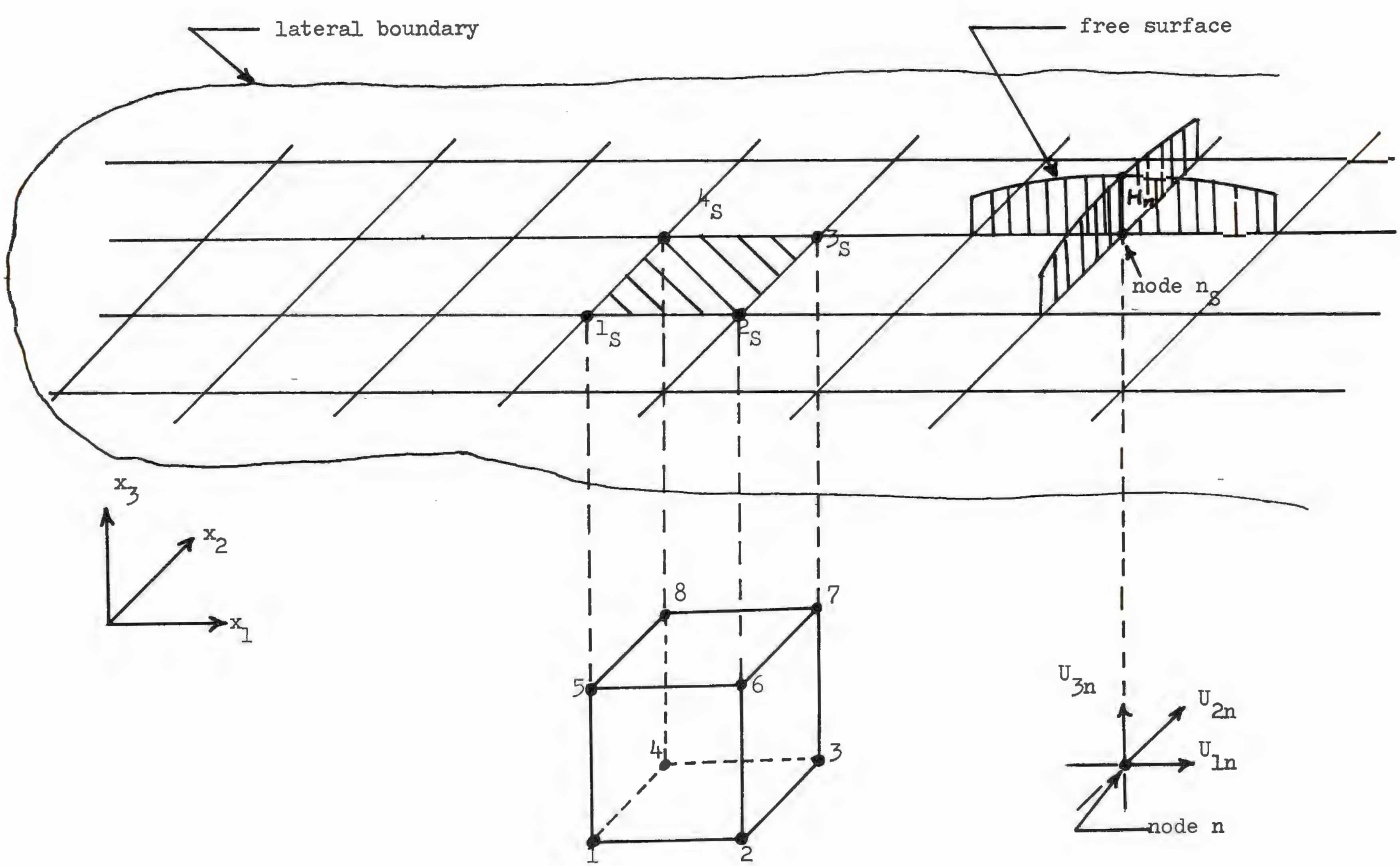

Figure 1. Typical Element in Lake Basin

$$\eta^* = \frac{1}{b_1 b_2} [\psi_o][A_o] \quad H_i \tag{15b}$$

where b_i = length of element in x_i direction, ψ = approximating polynomial over the volume element, ψ_o = approximating polynomial over the surface element, U_i = nodal values of u_i, H = nodal values of η, A and A_o = matrices of coefficients (dependent on nodal coordinates and choice of ψ) guaranteeing continuity of u_i and η across element interfaces. The products $\psi\ A/b_3$ and $\psi_o\ A_o/b_1 b_2$ correspond to the weighting functions ϕ and ϕ_o in Eqs. (10).

The matrix form of Eqs. (12) are

$$\sum_{e_v} \frac{1}{b_3} \int_{V_e} A^T \psi^T R_i \, dV_e = 0 \tag{16a}$$

$$\sum_{e_S} \frac{1}{b_1 b_2} \int_{S_e} A_o^T \psi_o^T R_o \, dS_e = 0 \tag{16b}$$

where e_v denotes the total number of volume elements, and e_S denotes the total number of surface elements. Therefore, after substitution of Eqs. (15) into Eqs. (16) and (14) and after regrouping of terms, the matrix form of Eqs. (14) are found to be

$$\sum_{e_v} \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ H \end{bmatrix} - \begin{bmatrix} F_1 \\ F_2 \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{17a}$$

$$\sum_{e_S} ([D_{41} \quad D_{42} \quad D_{43} \quad D_{44}]) \begin{bmatrix} U_{1S} \\ U_{2S} \\ U_{3S} \\ H \end{bmatrix} = 0 \tag{17b}$$

where (i = 1,2 and j = 1,3)

$$[D_{ij}] = \frac{1}{b_3^2}\,\epsilon\,[A]^T \int_{V_e} (\delta_{ij}[\psi_{,k}]^T[\psi_{,k}] + [\psi_{,j}]^T[\psi_{,i}] - \frac{f e_{ij}}{\epsilon}[\psi]^T[\psi])\,dV_e[A] \quad (18a)$$

$$[D_{3j}] = \frac{1}{b_3^2}[A]^T \int_{V_e} [\psi_{,j}]^T[\psi]\,dV_e[A] \quad (18b)$$

$$[D_{i4}] = -\frac{g[A]^T}{b_1 b_2 b_3} \int_{V_e} [\psi_{,i}]^T[\psi_o]\,dV_e[A_o] \quad (18c)$$

$$[D^{4i}] = [0] \quad (18d)$$

$$[D_{34}] = [0] \quad (18e)$$

$$[D_{43}] = \frac{1}{b_1^2 b_1^2}[A_o]^T \int_{S_e} [\psi_o]^T[\psi_o]\,dS_e[A_o] \quad (18f)$$

$$[D_{44}] = \frac{-1}{b_1^3 b_2^3}[A_o]^T \int_{S_e} [\psi_o]^T[\psi_o][A_o](\,U_{1S}\;[\psi_{o,1}] + U_{2S}\;[\psi_{o,2}])\,dS_e[A_o] \quad (18g)$$

$$F_i = \frac{1}{b_3}[A]^T \int_{S_e} [\psi]^T \frac{\tau_i^e}{\rho}\,dS_e - \frac{g}{b_3}[A]^T \int_{V_e} x_3\,[\psi_{,i}]^T dV_e \quad (18h)$$

$$Q = \sum_{k=1}^{\text{no. faces}} \frac{[A]^T}{b_3} \int_{S_k} [\psi]^T \frac{q_k^e}{\rho}\,dS_e \quad (18i)$$

where in Eq. (18h) $\tau_i = 0$ if the element in question is not at the surface of the lake, or τ_i = wind stress on the upper face of an element at the surface of the lake. In Eq. (18i), the summation extends over all faces of the element and $q_k = 0$ if the kth face is not an inlet/outlet region, or q_k = flow distribution across the kth face of the element at an inlet/outlet region.

Before the element matrices given above can be evaluated, particular choices of $[\psi]$ and $[A]$ must be made. In this instance a trilinear polynomial was assumed in the form

$$\psi = [\psi_o , x_3 \psi_o] \tag{19a}$$

$$\psi_o = [1, x_1, x_1 x_2, x_2] \tag{19b}$$

and a node numbering scheme as shown in Fig. 1 was assumed. Therefore,

$$A = \frac{1}{b_1 b_2} \left[\begin{array}{c|c} x_3^+ A_o & -x_3^- A_o \\ \hline -A_o & A_o \end{array}\right] \tag{20a}$$

$$A_o = b_1 b_2 \begin{bmatrix} 1 & x_1^- & x_1^- x_2^- & x_2^- \\ 1 & x_1^+ & x_1^+ x_2^- & x_2^- \\ 1 & x_1^+ & x_1^+ x_2^+ & x_2^+ \\ 1 & x_1^- & x_1^- x_2^+ & x_2^+ \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} x_1^+ x_2^+ & -x_1^- x_2^+ & x_1^- x_2^- & -x_1^+ x_2^- \\ -x_2^+ & x_2^+ & -x_2^- & x_2^- \\ 1 & -1 & 1 & -1 \\ -x_1^+ & x_1^- & -x_1^- & x_1^+ \end{bmatrix} \tag{20b}$$

where $x_i^{\pm} = x_i$-coordinate $(\pm)$ face. Substituting Eqs. (9) and (20) into Eqs. (18) and integrating, one obtains

$$D_{ij} = \left[\begin{array}{c|c} 2[M_{kk}\delta_{ij} + M_{ji} - (\frac{f}{\epsilon} e_{ij} - \frac{3}{b_3^2}\delta_{ij}) M_{oo}] & [M_{kk}\delta_{ij} + M_{ji} - (\frac{f}{\epsilon} e_{ij} + \frac{6}{b_3^2}\delta_{ij}) M_{oo}] \\ \hline [M_{kk}\delta_{ij} + M_{ji} - (\frac{f}{\epsilon} e_{ij} + \frac{6}{b_3^2}\delta_{ij}) M_{oo}] & 2[M_{kk}\delta_{ij} + M_{ji} - (\frac{f}{\epsilon} e_{ij} - \frac{3}{b_3^2}\delta_{ij}) M_{oo}] \end{array}\right] \tag{21a}$$

$$i,j = 1,2$$

$$D_{i3} = \frac{3}{b_3} \left[\begin{array}{c|c} -M_{oi} & -M_{oi} \\ \hline M_{oi} & M_{oi} \end{array}\right] \qquad i = 1,2 \tag{21b}$$

$$F_i = \frac{6b_1 b_2}{b_3^2} \; \frac{\bar{\tau}_i}{\epsilon\rho} \left[\frac{x_3^+ \quad M_o}{-x_3^- \quad M_o} \right] \tag{21c}$$

$$D_3^i = \left[\begin{array}{c|c} 2M_{io} & M_{io} \\ \hline M_{io} & 2M_{io} \end{array} \right] \qquad i = 1,2 \tag{21d}$$

$$D_{33} = \frac{3}{b_3} \left[\begin{array}{c|c} -M_{oo} & -M_{oo} \\ \hline M_{oo} & M_{oo} \end{array} \right] \tag{21e}$$

$$D_{i4} = \frac{-3g}{\epsilon} \left[\frac{M_{io}}{M_{io}} \right] \qquad i = 1,2 \tag{21f}$$

$$D_{43} = [M_{oo}] \tag{21g}$$

$$D_{44} = \frac{1}{b_1 b_2} \Big[[\ 0\ ] \Big| [M_{oo}]\ U_{15} \Big| [M_2]\ U_{15} + [M_1]\ U_{25} \Big| [M_{oo}]\ U_{25}\ ][A_o] \Big] \tag{21h}$$

$$Q = 3b_1 b_2/\rho \sum_k \bar{q}_k \left[R_1^{\pm} + R_2^{\pm} \right] \tag{21i}$$

where $\bar{\tau}_i$ and $\bar{q}$ denote average values of stress and flow over the face of the element, and M_{ij} and M_i $(i,j = 0,1,2)$ are given by

$$M_{ij} = \frac{1}{b_1 b_2} [A_o]^T \int_{S_e} [\psi_{o,i}]^T [\psi_{o,j}] dx_1 dx_2 [A_o] \tag{22a}$$

$$M_o = \frac{1}{b_1 b_2} [A_o]^T \int_{S_e} [\psi_o]^T ds_1 ds_2 \tag{22b}$$

$$M_i = \frac{1}{b_1 b_2} [A_o]^T \int_{S_e} x_i [\psi_o]^T [\psi_o] ds_1 ds_2 [A_o] \tag{22c}$$

$$R_i^{\pm} = \frac{1}{b_1 b_2} [A_o]^T \int_{x=x_i^{\pm}} [\psi_o]^T \, dx_{3-i} \qquad (22d)$$

where $x_i^{\pm}$ denotes the x_i-coordinate, x_i^+ or x_i^-, of the flow face, and where $\psi_{o,o} = \psi_o$. The integrals in Eqs. (22) are given in Table 1.

Once Eqs. (17) have been obtained for the typical element, the individual submatrices being defined by Eqs. (21), it is possible to generate a single master matrix equation for the total lake using assembly techniques developed for problems in solid mechanics (5). This has not been done in this instance because of the nonlinear nature of Eq. (17b), i.e., observe that D_{44} given by Eq. (21h) is a function of nodal velocities. Instead, the assemblage is postponed until the governing equations of the solution procedure have been developed.

SOLUTION TECHNIQUE

To generate numerical solutions to the governing nonlinear algebraic equations developed in the preceding section, an iterative solution technique is required. The technique described in this section is a well-known convergence acceleration scheme--the Newton-Raphson method.

In essence, the Newton-Raphson method (9) consists of obtaining the improvement ϵ^r to an initial guess u^r by means of the equation

$$\epsilon^r = -[J(u^r)] \, f(u^r) \qquad (23a)$$

where $f(u) = 0$ are the governing nonlinear equations, and

$$J_{ij}(u^r) = \frac{\partial f_i(u^r)}{\partial u_j^r} \qquad (23b)$$

$\int \psi_o{}^T\psi_o\, 1/b_1b_2$				$\int\psi_o{}^T\psi_{o,1}/b_1b_2$				$\int\psi_o{}^T\psi_{o,2}/b_1b_2$			
1	$\bar{x}_1$	$\overline{x_1x_2}$	$\bar{x}_2$	0	1	$\bar{x}_2$	0	0	0	$\bar{x}_1$	1
$\bar{x}_1$	$\bar{\bar{x}}_1$	$\overline{\bar{x}_1x_2}$	$\overline{x_1x_2}$	0	$\bar{x}_1$	$\overline{x_1x_2}$	0	0	0	$\bar{\bar{x}}_1$	$\bar{x}_1$
$\overline{x_1x_2}$	$\overline{\bar{x}_1x_2}$	$\overline{\bar{x}_1\bar{x}_2}$	$\overline{x_1\bar{x}_2}$	0	$\overline{x_1x_2}$	$\overline{x_1\bar{x}_2}$	0	0	0	$\overline{\bar{x}_1x_2}$	$\overline{x_1x_2}$
$\bar{x}_2$	$\overline{x_1x_2}$	$\overline{x_1\bar{x}_2}$	$\bar{\bar{x}}_2$	0	$\bar{x}_2$	$\bar{\bar{x}}_2$	0	0	0	$\overline{x_1x_2}$	$\bar{x}_2$
$\int\psi^T_{o,1}\,\psi_{o,1}/b_1b_2$				$\int\psi^T_{,1}\,\psi_{o,2}/b_1b_2$				$\int\psi^T_{,2}\,\psi_{,2}/b_1b_2$			
0	0	0	0	0	0	0	0	0	0	0	0
0	1	$\bar{x}_2$	0	0	0	$\bar{x}_1$	1	0	0	0	0
0	$\bar{x}_2$	$\bar{\bar{x}}_2$	0	0	0	$\overline{x_1x_2}$	$\bar{x}_2$	0	0	$\bar{\bar{x}}_1$	$\bar{x}_1$
0	0	0	0	0	0	0	0	0	0	$\bar{x}_1$	1
$\int\psi_o/b_1b_2$				$\int x_1\psi_o^T\psi_o/b_1b_2$				$\int x_2\psi_o^T\psi_o/b_1b_2$			
1	$\bar{x}_1$	$\overline{x_1x_2}$	x_2	$\bar{x}_1$	$\bar{\bar{x}}_1$	$\overline{\bar{x}_1x_2}$	$\overline{x_1x_2}$	$\bar{\bar{x}}_2$	$\overline{x_1x_2}$	$\overline{x_1\bar{x}_2}$	$\bar{\bar{x}}_2$
$\bar{x}_i=(x^+_i+x^-_i)/2$				$\bar{\bar{x}}_1$	$\bar{\bar{\bar{x}}}_1$	$\overline{\bar{\bar{x}}_1x_2}$	$\overline{\bar{x}_1x_2}$	$\overline{x_1x_2}$	$\overline{\bar{x}_1x_2}$	$\overline{\bar{x}_1\bar{x}_2}$	$\overline{x_1\bar{x}_2}$
$\bar{\bar{x}}_i=x^+_ix^-_i+b_i^2/3$				$\overline{\bar{x}_1x_2}$	$\overline{\bar{\bar{x}}_1x_2}$	$\overline{\bar{\bar{x}}_1\bar{x}_2}$	$\overline{\bar{x}_1\bar{x}_2}$	$\overline{x_1\bar{x}_2}$	$\overline{\bar{x}_1\bar{x}_2}$	$\overline{\bar{x}_1\bar{\bar{x}}_2}$	$\overline{x_1\bar{\bar{x}}_2}$
$\bar{\bar{\bar{x}}}_i=\bar{x}_i(x_i^{+2}+x_i^{-2})/2$				$\overline{x_1x_2}$	$\overline{\bar{x}_1x_2}$	$\overline{\bar{x}_1\bar{x}_2}$	$\overline{x_1\bar{x}_2}$	$\bar{\bar{x}}_2$	$\overline{x_1\bar{x}_2}$	$\overline{x_1\bar{\bar{x}}_2}$	$\bar{\bar{\bar{x}}}_2$

Table 1. Integrals in Eqs. (22)($[M_{ij}]=[M^T_{ji}]$)

The new approximation is then given by

$$u^{r+1} = u^r + \epsilon^r \qquad (23c)$$

Therefore, given an initial guess, u^o, repeated application of Eqs. (23) will yield successively better approximations to the true solution.

The primary tasks in the application of the Newton-Raphson method to the current problem are to generate [J] and to obtain an initial guess in the neighborhood of the true solution. For the typical element the governing equations are given by Eqs. (17). Applying Eq. (23b), one obtains for each element

$$\sum_{e_v} \left(\begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \end{bmatrix} \begin{bmatrix} \epsilon_{u_1} \\ \epsilon_{u_2} \\ \epsilon_{u_3} \\ \epsilon_H \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2^r \\ u_3^r \\ H^r \end{bmatrix} - \begin{bmatrix} F_1 \\ F_2 \\ Q \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (24a)$$

$$\sum_{e_S} \left(\begin{bmatrix} J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix} \begin{bmatrix} \epsilon_{u_{1S}} \\ \epsilon_{u_{2S}} \\ \epsilon_{u_{3S}} \\ \epsilon_H \end{bmatrix} + \right.$$

$$+ \quad [D_{41} \quad D_{42} \quad D_{43} \quad D_{44}] \begin{bmatrix} u^r_{1S} \\ u^r_{2S} \\ u^r_{3S} \\ H^r \end{bmatrix} \Bigg) = 0 \tag{24b}$$

where

$$J_{ij} = D_{ij} + \Delta J_{ij} \tag{24c}$$

$$\Delta J_{ij} = 0 \qquad i = 1,2,3;\ j = 1,2,3,4 \tag{24d}$$

$$\Delta J_{4j} = 0 \qquad j = 3,4 \tag{24e}$$

$$\Delta J_{4j} = \frac{1}{b_1 b_2}\Big[[M_{oo}]\cdot([A_{2\cdot j}]\ H^r\ ) + [M_{3-j}]\cdot([A_3]\ H^r\ ])\Big] \qquad j = 1,2 \tag{24f}$$

In Eq. (24f) A_i = ith row of matrix A given by Eq. (20b) and $[B]\cdot(f)$ denotes multiplication of the matrix B by the scalar f.

If it is assumed for the moment that an initial guess U^o_i, H^o at every node is readily available, then the assemblage and solution scheme could be as follows:

1) Calculate all D_{ij} for element e. (If this element adjoins the free surface calculate all D_{4j} except D_{44}).
2) For a surface element use appropriate subvectors of U^o_i and H^o to calculate D^o_{44} and the elemental vector

$$P^o = \left[\begin{array}{c|c} D_{ij} & D_{i4} \\ \hline D_{4i} & D^o_{44} \end{array}\right] \left[\begin{array}{c} U^o_i \\ \hline H^o \end{array}\right] - \left[\begin{array}{c} F_i \\ \hline 0 \end{array}\right] \tag{25}$$

3) Add subvectors of P^o to appropriate locations of master matrix P^o_T arranged by nodal groupings of variables.

4) Generate all J^o_{ij} for element e using Eq. (24c) and add submatrices to appropriate locations of master matrix $[J^o_T]$ also arranged by nodal groupings of variables (note that only J^o_{41} and J^o_{42} differ from values stored in submatrices of D and that J_{i4} are associated with the corresponding surface nodes for element e).

5) Account for boundary conditions by setting the forcing function to zero at boundary nodes and by replacing the rows and columns of J^o_S associated with boundary nodes by rows and columns of zeros with an identity matrix in the diagonal position.

6) Solve for the correction vector ϵ_T,

$$\epsilon_T = [J^o_T]^{-1} \; P^o_T \tag{26}$$

and update the most recent trial solution.

7) If ϵ_T is sufficiently small, output the final solution. If ϵ_T is not sufficiently small, repeat steps 2 through 6.

The problem remains of determining an initial guess, U^o_i and H^o. Since the governing equations are only slightly nonlinear, the equivalent linear solution to Eqs. (17) can be used as an initial guess. The matrix D_{i4} is suppressed and Eq. (17a) yields a system matrix of the form

$$\left[\begin{array}{c|c} D_{ij_T} & 0 \\ \hline D_{4i_T} & D_{44_T} \end{array}\right] \left[\begin{array}{c} U_{i_T} \\ \hline H_T \end{array}\right] = \left[\begin{array}{c} F_{i_T} \\ \hline 0 \end{array}\right] \tag{27}$$

The solution to Eq. (27) can be seen to be

$$U_{i_T} = [D_{ij_T}]^{-1} \; F_{i_T}$$

$$H = [D_{44_T}(U_i)]^{-1}[D_{4i}]_T U_{i_T}$$

which are then taken as the initial guesses for the solution process outlined above.

The selection procedure for an initial guess can in fact be incorporated easily as a substep between Steps 1 and 2 above. For the first iteration: 1) set U^o and H equal to zero; 2) suppress D_{i4}, D_{4i}, and J_{4i}; 3) set J_{44} and D_{44} equal to identity matrices; 4) solve for ϵ_T and update; 5) insert the proper values of D_{i4}, D_{4i}, D_{i4}, J_{i4} and J_{4i}; 6) set $J_{4i} = D_{i4}$ and resolve for ϵ_T; and finally, 7) rejoin solution process given above at Step 2.

SUMMARY

A finite element model of the massive circulation in shallow lake basins has been formulated based on the method of weighted residuals. Unlike previous analytic and finite element models, the three-dimensional character of the basin was accounted for by using rectangular parallelopiped elements with associated surface nodes.

The governing field equations assumed in the derivation of the model are those due to Ekman (1) in which vertical momentum transfer is neglected compared to gravitational forces implying a hydrostatic pressure distribution. Also, inertial forces are neglected compared to the coriolis and surface wind forces.

The field equations and their corresponding finite element analogues are linear. However, the auxiliary equations expressing the free surface boundary conditions is nonlinear in nature. Therefore, an iterative solution technique, the Newton-Raphson method, is the

basis for solution to the algebraic matrix equation. An outline of this method as applied to the problem at hand is given along with a scheme for the automatic generation of an initial trial solution required for the Newton-Raphson method.

Numerical results to verify and demonstrate the formulations presented herein are as yet unavailable. A computer program to obtain such results has been written and is currently being debugged. One difficulty anticipated in verifying the model is that there are no exact, or even approximate, results available for three-dimensional flow in a lake basin. Only depth-averaged two-dimensional flows (with inherent inaccuracies) have been considered. With such models only relative rather than absolute values for the free surface elevation are obtainable. Also, experimental data suitable for comparison purposes are scarce and scattered.

It is feasible to consider various extensions of the finite element model presented herein. Steady-state flow in thermally stratified lakes can be incorporated by allowing different values for the pertinent fluid properties in different elements. Transient flow capabilities can be added to the linearized model by retaining the linear portion of the convective derivative in the Navier-Stokes equation. If the assumption of negligible inertial forces is not used, the fully nonlinear field equations can be modeled by a finite element analogue as described in Reference 10. Such a model would be useful to describe the complicated flow in the regions of inlet or outlet rivers where in fact the Coriolis and surface wind forces are negligible compared to the inertial forces rather than the converse as assumed herein.

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