STUDY OF THE RHEOLOGY OF AN ELASTIC MEDIUM THROUGH THE SPLITTING OF ITS EIGENFREQUENCIES.

Michele Capiuto

Dipartimento di Fisica, Università "La Sapienza", Roma, Italy.

Abstract.

It is seen how the rheology causes the splitting of the eigenfrequencies of a harmonic elastic oscillator into a number of lines spread over a frequency band depending on the rheology of the medium. The observation of these lines may allow to infer properties of the rheology.

In most cases the splitting of the eigenvalues of a system of partial differential equations is due to the removal of some geometric symmetry in the model considered.

In the model of this note the splitting is due to the removal of the symmetry of the time coordinate since the presence of the dissipation makes the phenomenon irreversible.
Introduction.

Among the phenomena occurring in Physics the dissipation of energy is of great interest in many branches of applied research ranging from general mechanics to seismic constructions.

The effect, for a simple oscillator, is generally measured with the quality factor \( Q \) defined as the ratio

\[
Q^{-1} = \frac{\Delta E}{2\pi E}
\]

where \( E \) is the peak energy and \( \Delta E \) is the decrease in energy between two successive peaks or the percenttual loss of energy per unit cycle.

From the experimental point of view the phenomenon results in a broadening of the spectral lines of the spectrum.

But the phenomenon may be seen also from another point of view relating it to the eigenvalues of the oscillator.

We shall see here that in the case of an anelastic medium in which the stress-strain relations contain a memory mechanism represented by a convolution as follows

\[
h \ast \varepsilon^{(m+1)} + \mu \varepsilon = \mu h \ast \varepsilon^{(m+1)}
\]

where \( \varepsilon \) is stress, \( \varepsilon \) is strain, \( \mu \) is the elastic parameter, and \( h(t) \) represents the memory mechanism acting through the
convolution indicated by \* . For a wide class of elastic materials \( h(t) \) may be assumed (Caputo 1967)

\[
h(t) = \eta \frac{E^2}{\Gamma(1-z)} , \quad H(p) = L[T[h(t)]] = \eta p^{2-z}
\] (3)

In that case the relation (2) becomes a generalization of that of Maxwell in which the derivative of first order with respect to time is substituted by that of real order \( m + z \).

It was also shown that the stress-strain relation (2) with \( h(t) \) as in (3) may represent also the phenomenon of fatigue (Caputo 1979) by considering the hysteresis loop of the medium subject to a cyclic deformation; it allows to compute the number of cycles which would give the fatigue as function of the frequency and of the amplitude of the cyclic strain applied.

The solution for the harmonic oscillator.

We shall see here that the stress-strain relation (2) with \( h(t) \) as in (3), causes a splitting of each of the spectral lines of the oscillator in a set of lines depending on the value of \( z \) and limited in \( m \) and whose width depends on \( \eta, m, z \) and on the frequency.

In fact, indicating by \( u(x,t) \) the displacement of a point of the medium and assuming that this, for \( t = 0 \), is at rest, by taking the Laplace Transform (LT) of (2)
and substituting in the equilibrium condition we find

\[
\eta p^{m+z} T(p, x) + \mu T(p, x) = \eta \mu E(p, x) p^{m+z}
\]

(4)

\[
\rho \rho^2 U = \frac{\mu \eta p^{m+z}}{\mu + \eta p^{m+z}} \frac{\partial^2 U}{\partial x^2}
\]

where capital letters indicate the LT of the functions indicated by the corresponding lower case letter and \( p \) is the LT variable; \( \rho \) is the density.

To obtain (4) we used the theorem of Appendix A of Caputo (1969) which extends to the derivatives of real order the well known theorem of the LT of derivatives of integer order.

Equation (4) gives

\[
U = A(p) e^{w x} + B(p) e^{-w x}
\]

(5)

\[
w^2 = \rho \rho^2 (\mu + \eta p^{z+m})/\mu \eta p^{z+m}
\]

As boundary conditions we may choose those when the displacement is nil for any \( t \) at \( x = 0 \) and \( x = s \). Which imply that

\[
A + B = 0
\]

\[
e^{w s} - e^{-w s} = 0
\]

(6)
Substituting \( p = i \omega \) in (6), with \( \omega \) angular frequency, we have

\[
\sin \left[ (a + ib)^{1/2} \omega \sqrt{1/p} \right] = 0
\]

\[
a(\omega) = 1 + \frac{\mu}{\eta} \omega^{z-m} \cos\left[ \frac{\pi}{2} (z + m) \right]
\]

\[
b(\omega) = -\frac{\mu}{\eta} \omega^{z-m} \cos\left[ \frac{\pi}{2} (z + m) \right]
\]

The eigenvalues are obtained solving the equation

\[
\rho (a + ib) \omega^2 = \mu n^2 r^2
\]

It is seen that for every \( n \) there is more than one eigenvalue due to the presence of \( k \). They are

\[
\omega_n / \Re \left[ a(\omega_n) + i b(\omega_n) \right]
\]

\[
\omega_n = \frac{\mu \pi}{\rho} \sqrt{\frac{\pi}{2} / \mu}
\]

For many elastic media we may reasonably assume that \( \eta \omega^{m+2} / \mu \gg 1 \), then

\[
\omega = \omega_n \left( 1 - \frac{\mu \omega_n^{z-m}}{2 \eta} \cos\left[ \frac{\pi}{2} (z + m) \right] \right)
\]

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It is thus clear that depending on the value of \( z \) one may have a finite or an infinite number of eigenvalues for each \( n \).

The \( Q^{-1} \) of the spectral lines is one half the ratio of the imaginary to the real part of \( \omega \)

\[
Q^{-1} = \frac{\mu}{2\eta} \omega_n^{-z-m} \sin \left[ (\frac{\pi}{2} + 2k\pi)(z + m) \right]
\]  

(11)

The solution, for each value of \( n \), are limited in the band

\[
\omega_n (1 - \omega_n^{-z-m} \mu/2\eta), \quad \omega_n (1 + \omega_n^{-z-m} \mu/2\eta)
\]  

(12)

The term \( m(1/2 + 2k)\pi \) does not affect the number of eigenvalues, also, since \( n \) does not enter the corrective factor of (10), it is clear that each eigenvalue, corresponding to a value of \( n \) of the perfectly elastic case, is split in the same number of eigenvalues, for all \( z \), which depends only on \( z \).

The effect of \( m \) depends only on its parity and is limited to a shift of the solutions, in fact is \( m \) is even and multiple of four then the argument of the cosine is shifted by a multiple of \( 2\pi \), if \( m \) is even and not multiple of four then the argument of the cosine is shifted by \( \pi \), if \( m \) is even but not multiple of four then the
argument of the cosine is shifted by $\pi/2$.

If $z$ is not rational then there is an infinite number of eigenvalues limited in the band defined by (12). If $z$ is rational the number of eigenvalues is limited.

In the following we shall consider few cases in which $z$ is rational which will illustrate how the solutions are spread in the range (12) and discuss the corresponding values of $Q$.

The number of eigenvalues generated in the splitting.

It is no limitation to the discussion to assume that $m$ is a multiple of four, that $z = q/(4l + 1)$ with $q$ and $l$ integer and also that $q$, $4l + 1$ are relatively prime.

When $q = 4r + 1$, with $r$ integer the values of $\cos(1/2 + 2k) z$, as function of $k$ in the range $-21, 21$, are repeated periodically when $|k| > 21$; the resulting eigenvalues (10) are therefore $4l + 1$ and correspond to an equal number of spectral lines.

When $q = 2r + 1$, with $r$ odd, one may see that the spectral lines are again $21 + 1$. When $q$ is even, due to the symmetry of the cosine and to the symmetry of its argument with respect to $k = \pm 1, \pm 3, \pm 5, \ldots$ the values of $\cos(1/2 + 2k) z$, as function of $k$, in the range $-1, 1$ are repeated when $|k| > 1$; the resulting eigenvalues (10) are therefore $21 + 1$ and correspond to an equal number of spectral lines.
The values of $Q$ corresponding to each of the spectral lines is given by (11); depending on the argument of the sine we may therefore have modes without dissipation, they correspond to frequencies on the extremes of the band defined by (12).

The spectral lines with the largest $Q^{-1}$ are those for which the argument of the sine is $(2u + 1)\pi/2$ with $u$ even; they correspond to the eigenfrequencies of the purely elastic case.

The eigenvalues which correspond to the argument of sine with $u$ odd would have a negative $Q$ and are therefore disregarded for their physical insignificance.

The discussion of the cases when $z = q/(2l + 1)$, with $l$ odd, or $z = q/2l$ is similar to that made for the case when $z = q/(4l + 1)$.

Conclusions

It is seen how the rheology causes a splitting of the spectral lines of an oscillator.

In general the rheology of elastic media is studied observing phenomena with very low forced frequency therefore requiring that the observations are taken for very long
time, which is generally difficult to obtain. The observation of the splitting of the spectral lines may allow to infer properties of the rheology with observations taken for a time interval one order of magnitude longer than the length of the period of the fundamental mode.

References.


M. Caputo, Elasticità e dissipazione (Zanichelli, Bologna), Chap. 2, pp. 39-71; Chap. 3, pp. 73-114.


Table caption

Values of the cosine of formula (10) giving the eigenvalues for \( z = q/(21 + 1) \), \( m = 0 \) with \( 1 = 3 \) and \( q = 1, 2, 3, 4, 5, 6 \). When \( q \) is even there are 4 different eigenvalues when \( q \) is odd there are 7 different eigenvalues.
### Table

#### Values of $q$

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#### Eigenvalues

| -1.00 | -0.97 | -0.90 | -0.78 | -0.62 | -0.43 | -0.22 | 0 |

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#### Eigenvalues

| 0.22 | 0.43 | 0.62 | 0.78 | 0.90 | 0.97 | 1.0 |